

Quasi-Elliptic Cohomology and its Spectra

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ABSTRACT. Goerss, Hopkins and Miller have proved that the moduli stack of elliptic curves can be covered by E_∞ elliptic spectra. It is not known whether this result can be extended to global elliptic cohomology theories and global ring spectra. Generally, it's difficult to construct the representing spectra of an elliptic cohomology. In this paper we construct an orthogonal G -spectrum for each compact Lie group G which weakly represents quasi-elliptic cohomology. Unfortunately, our construction does not arise from a global spectra; thus, in a coming paper we consider a new formulation of global stable homotopy theory that contains quasi-elliptic cohomology.

1. Introduction

An elliptic cohomology theory is a generalized cohomology theory corresponding to an elliptic curve. As shown in [41], the moduli stack of elliptic curves can be covered by E_∞ elliptic spectra. A good question is whether this conclusion can be extended to global elliptic cohomology.

It's an old idea of Witten, as shown in [38], that the elliptic cohomology of a space X is related to the \mathbb{T} -equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$, with the circle \mathbb{T} acting on LX by rotating loops. A new theory quasi-elliptic cohomology is introduced to interpret the relation between Tate K-theory and the loop spaces. Quasi-elliptic cohomology $QEll^*$ is closed to Tate K-theory, the generalized elliptic cohomology associated to the Tate curve. But unlike most elliptic cohomology theories, it has a neat form and can be expressed explicitly by equivariant K-theories.

$$(1.1) \quad QEll_G^*(X) := \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda(\sigma)}^*(X^\sigma) = \left(\prod_{\sigma \in G^{tors}} K_{\Lambda(\sigma)}^*(X^\sigma) \right)^G$$

which can be constructed from equivariant K-theories. The definition (1.1) is fully explained in [34] and [51].

As equivariant K-theories, it has the change of group isomorphism. Ganter indicated that $\{QEll_G^*\}_G$ for all the compact Lie groups can be put together naturally in a uniform way and make an ultra-commutative global cohomology theory. In this paper we construct G -equivariant spectra of the theory, and then construct G -orthogonal spectra weakly representing it.

2010 *Mathematics Subject Classification.* Primary 55.

The author was partially supported by NSF grant DMS-1406121.

However, we show that the G -orthogonal spectra cannot arise from an orthogonal spectrum. Instead, in a coming paper we construct a new global homotopy theory and show there is a global orthogonal spectrum in this new global homotopy theory that weakly represents orthogonal quasi-elliptic cohomology.

1.1. Equivariant spectra. The main conclusion of this part is below.

THEOREM 1.1. *For each compact Lie group G and each integer n , there is a space $QEll_{G,n}$ representing $QEll_G^n(-)$ in the sense of (1.2).*

$$(1.2) \quad \pi_0(QEll_{G,n}) = QEll_G^n(S^0), \text{ for each } k.$$

Let $KU_{G,n}$ denote the space representing the n -th G -equivariant K -theory. First we construct in Theorem 4.6 a homotopical right adjoint R_g for the functor $X \mapsto X^g$ from the category of G -spaces to the category of $\Lambda_G(g)$ -spaces. Then we get

$$\prod_{g \in G_{conj}^{tors}} \text{Map}_{\Lambda_G(g)}(X^g, KU_{\Lambda_G(g),n})$$

is weakly equivalent to

$$\text{Map}_G(X, \prod_{g \in G_{conj}^{tors}} R_g(KU_{\Lambda_G(g),n})),$$

as stated in Theorem 4.7.

So $QEll_{G,n} := \prod_{g \in G_{conj}^{tors}} R_g(KU_{\Lambda_G(g),n})$ is one choice of the classifying space we want.

1.2. G -orthogonal spectra. Based on the construction of $QEll_{G,n}$, we construct for each faithful G -representation V a space $E(G, V)$ that weakly represents $QEll_G^V(-)$ in the sense of (1.3),

$$(1.3) \quad \pi_k(E(G, V)) = QEll_G^V(S^k), \text{ for each } k.$$

Moreover, we can construct the structure maps for E and obtain the main conclusion of this paper.

THEOREM 1.2. *There is an orthogonal G -spectrum and \mathcal{I}_G -FSP weakly representing quasi-elliptic cohomology.*

In addition, we construct the restriction maps $E(G, V) \rightarrow E(H, V)$ for each group homomorphism $H \rightarrow G$. This map is not a homeomorphism, but an H -weak equivalence.

The orthogonal G -spectra $E(G, -)$ cannot arise from an orthogonal spectrum. This fact motivates us to construct a new global homotopy theory in a coming paper.

REMARK 1.3. The construction of the G -FSP E is not related to the concrete construction of the global K-theory, which means, for any ultra commutative global ring spectrum X for a global cohomology theory H , we can form a \mathcal{I}_G -FSP that weakly represents the cohomology theory

$$X \mapsto \prod_{\sigma \in G_{conj}^{tors}} H_{\Lambda(\sigma)}^*(X^\sigma)$$

in a way analogous to the construction of E .

In Section 2 we recall the basics in equivariant homotopy theory and equivariant obstruction theory. In Section 3 we recall the definition of quasi-elliptic cohomology. In Section 4, we show the construction of the equivariant spectra of quasi-elliptic cohomology. In Section 5 we recall the basics of global homotopy theory and the construction of global K-theory. In Section 6 we construct an orthogonal G -spectra for the theory, which is a \mathcal{I}_G -FSP.

Acknowledgement. I thank Charles Rezk for suggesting the idea of homotopical adjunction and weak spectra. He initiated the project and supported my work all the time. I also thank Matthew Ando, Stefan Schwede, and Nathaniel Stapleton for helpful remarks.

2. Basics in equivariant homotopy theory

In this section we give a brief introduction of the equivariant homotopy theory and introduce the basic notions and concepts. The main references are [12], [20] and [45].

Let G be a compact Lie group. Let \mathcal{T} denote the category of topological spaces and continuous maps. Let $G\mathcal{T}$ denote the category of G -spaces, namely, spaces X equipped with continuous G -action

$$G \times X \longrightarrow X$$

and continuous G -maps.

Let H be a closed subgroup of G . Let X be a G -space and Y an H -space. Define

$$(2.1) \quad X^H := \{x | hx = x, \forall h \in H\}.$$

For $x \in X$, the isotropy group of x

$$(2.2) \quad G_x := \{h | hx = x\}.$$

The induced G -space

$$G \times_H Y$$

is a quotient space of $G \times Y$ with (gh, x) and (g, hx) identified for $g \in G$, $h \in H$. The coinduced G -space

$$\mathrm{Map}_H(G, Y)$$

is the space of H -maps $G \longrightarrow Y$ with a left action by G induced by the right action of G on itself, namely

$$(g \cdot f)(g') = f(g'g).$$

We have the adjunctions

$$(2.3) \quad G\mathcal{T}(G \times_H Y, X) \cong H\mathcal{T}(Y, X)$$

and

$$(2.4) \quad H\mathcal{T}(X, Y) \cong G\mathcal{T}(X, \mathrm{Map}_H(G, Y)).$$

DEFINITION 2.1. A G -homotopy between G -maps $X \rightrightarrows Y$ is a G -map

$$X \times I \longrightarrow Y$$

where $I = [0, 1]$ is a trivial G -space.

This gives as a homotopy category $hG\mathcal{T}$ whose objects are G -spaces and morphisms are G -homotopy classes of continuous G -maps. Recall that a map of spaces is a weak equivalence if it induces an isomorphism of all homotopy groups.

DEFINITION 2.2. A G -map $f : X_1 \rightarrow X_2$ is said to be weak equivalence if $f^H : X_1^H \rightarrow X_2^H$ is a weak equivalence for all the subgroups H of G .

Let $\bar{h}G\mathcal{T}$ denote the category constructed from $hG\mathcal{T}$ by adjoining formal inverses to the weak equivalences. $\bar{h}G\mathcal{T}$ is the desired homotopy category which contains all the algebraic invariants of G -spaces we are interested in.

THEOREM 2.3 (Elemendorf's Theorem). *The category $\bar{h}\mathcal{T}^{Op}_G$ and $\bar{h}G\mathcal{T}$ are equivalent.*

Let's also recall how equivariant CW-complex is constructed.

Let X be a space of the homotopy type of a G -CW complex. Let X_n denotes the n -skeleton of X . X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G -cells $G/H \times D^{n+1}$ along attaching G -maps $G/H \times S^n \rightarrow X^n$.

We have the homotopy pushout for each k

$$(2.5) \quad \begin{array}{ccc} \coprod G/H \times S^k & \longrightarrow & \coprod G/H \times D^k \\ \downarrow & & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$

And X is the homotopy colimit of the diagram

$$(2.6) \quad \begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & & \\ & & & & X & & \end{array}$$

with each map in it an inclusion.

Let GC denote the category of G -CW complexes and cellular maps. Proposition 2.4 is a conclusion needed for the construction later. It can be proved by induction.

PROPOSITION 2.4. Let D be a complete category. Let $i : \mathcal{O}_G^{Op} \rightarrow GC^{Op}$ be the inclusion of subcategory. If $F_1, F_2 : GC^{Op} \rightarrow D$ are two functors sending homotopy colimit to homotopy limit and if we have a natural transformation $p : F_1 \rightarrow F_2$, which gives a weak equivalence at orbits, then it also gives a weak equivalence on GC .

Especially, if p gives a retract at each orbit, F_1 is a retract of F_2 at each G -CW complexes.

3. Quasi-elliptic cohomology

In this section we recall the definition of quasi-elliptic cohomology. The main reference is [34] and [51] Before that we discuss in Section 3.1 the representation ring of $\Lambda_G(g)$.

3.1. Preliminary: representation ring of $\Lambda_G(g)$. For any compact Lie group G and a torsion element $g \in G$, Let $\Lambda_G(g)$ denote the group

$$\Lambda_G(g) = C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle$$

and \mathbb{T} the circle group \mathbb{R}/\mathbb{Z} . Let $q : \mathbb{T} \rightarrow U(1)$ be the isomorphism $t \mapsto e^{2\pi i t}$. The representation ring $R\mathbb{T}$ of the circle group is $\mathbb{Z}[q^\pm]$.

We have an exact sequence

$$1 \rightarrow C_G(g) \rightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \rightarrow 0$$

where the first map is $g \mapsto [g, 0]$ and the second map is

$$(3.1) \quad \pi([g, t]) = e^{2\pi i t}.$$

There is a relation between the representation ring of $C_G(g)$ and that of $\Lambda_G(g)$, which is shown as Lemma 1.2 in [51].

LEMMA 3.1. $\pi^* : R\mathbb{T} \rightarrow R\Lambda_G(g)$ exhibits $R\Lambda_G(g)$ as a free $R\mathbb{T}$ -module.

In particular, there is an $R\mathbb{T}$ -basis of $R\Lambda_G(g)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|_{C_G(g)}$ to $C_G(g)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(g)$.

PROOF. Let l be the order of the torsion element g . Note that $\Lambda_G(g)$ is isomorphic to

$$C_G(g) \times \mathbb{R}/l\mathbb{Z} / \langle (g, -1) \rangle.$$

Thus, it is the quotient of the product of two compact Lie groups.

Let $\lambda : C_G(g) \rightarrow GL(n, \mathbb{C})$ be an n -dimensional $C_G(g)$ -representation with representation space V and $\eta : \mathbb{R} \rightarrow GL(n, \mathbb{C})$ be a representation of \mathbb{R} such that $\lambda(g)$ acts on V via scalar multiplication by $\eta(1)$. Define

$$(3.2) \quad \lambda \odot_{\mathbb{C}} \eta([h, t]) := \lambda(h)\eta(t).$$

It's straightforward to verify $\lambda \odot_{\mathbb{C}} \eta$ is a n -dimensional $\Lambda_G(g)$ -representation with representation space V .

Any irreducible n -dimensional representation of the quotient group $\Lambda_G(g) = C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle$ is an irreducible n -dimensional representation of the product $C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle$. And any finite dimensional irreducible representation of the product of two compact Lie groups is the tensor product of an irreducible representation of each factor. So any irreducible representation of the quotient group $\Lambda_G(g)$ is the tensor product of an irreducible representation λ of $C_G(g)$ with representation space V and an irreducible representation η of \mathbb{R} . Any irreducible complex representation η of \mathbb{R} is one dimensional. So the representation space of $\lambda \otimes \eta$ is still V . Let l be the order of g . $\eta(1)^l = I$. We need $\eta(1) = \lambda(g)$. So $\eta(1) = e^{\frac{2\pi i k}{l}}$ for some $k \in \mathbb{Z}$. So

$$\eta(t) = e^{\frac{2\pi i (k+lm)t}{l}}.$$

Any $m \in \mathbb{Z}$ gives a choice of η in this case. And η is a representation of $\mathbb{R}/l\mathbb{Z} \cong \mathbb{T}$.

Therefore, we have a bijective correspondence between

(1) isomorphism classes of irreducible $\Lambda_G(g)$ -representation ρ , and

(2) isomorphism classes of pairs (λ, η) where λ is an irreducible $C_G(g)$ -representation

and $\eta : \mathbb{R} \rightarrow \mathbb{C}^*$ is a character such that $\lambda(g) = \eta(1)I$. $\lambda = \rho|_{C_G(g)}$. \square

REMARK 3.2. We can make a canonical choice of $\mathbb{Z}[q^\pm]$ -basis for $R\Lambda_G(g)$. For each irreducible G -representation $\rho : G \longrightarrow \text{Aut}(G)$, write $\rho(\sigma) = e^{2\pi ic} \text{id}$ for $c \in [0, 1)$, and set $\chi_\rho(t) = e^{2\pi ict}$. Then the pair (ρ, χ_ρ) corresponds to a unique irreducible $\Lambda_G(g)$ -representation.

3.2. Quasi-elliptic cohomology. In this section we introduce the definition of quasi-elliptic cohomology $QEll_G^*$ in term of orbifold K-theory, and then express it via equivariant K-theory. The readers may read Chapter 3 in [3] and [48] for a reference of orbifold K-theory.

Let X be a G -space. Let $G^{tors} \subseteq G$ be the set of torsion elements of G . Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)$ -space. We can define a $\Lambda_G(\sigma)$ -action on X^σ by

$$[g, t] \cdot x := g \cdot x.$$

Then quasi-elliptic cohomology is defined by

DEFINITION 3.3.

$$(3.3) \quad QEll_G^*(X) = \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}^*(X^g) = \left(\prod_{g \in G^{tors}} K_{\Lambda_G(g)}^*(X^g) \right)^G,$$

where G_{conj}^{tors} is a set of representatives of G -conjugacy classes in G^{tors} .

By computing the representation rings of $R\Lambda_G(g)$, we get $QEll_G^*(-)$ for contractible spaces. Then, using Mayer-Vietoris sequence, we can compute $QEll_G^*(-)$ for any G -CW complex by patching the G -cells together.

We have the ring homomorphism

$$\mathbb{Z}[q^\pm] = K_{\mathbb{T}}^0(\text{pt}) \xrightarrow{\pi^*} K_{\Lambda_G(g)}^0(\text{pt}) \longrightarrow K_{\Lambda_G(g)}^0(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection defined in (3.1) and the second is via the collapsing map $X \longrightarrow \text{pt}$. So $QEll_G^*(X)$ is naturally a $\mathbb{Z}[q^\pm]$ -algebra.

4. Equivariant Spectra

In this section, for each integer n , each compact Lie group G , we construct a space $QEll_{G,n}$ representing the n -th G -equivariant quasi-elliptic cohomology $QEll_G^n$ up to weak equivalence.

Before constructing the spectra, we explain what a good "weak equivalence" means.

DEFINITION 4.1 (homotopical adjunction). Let H and G be two compact Lie groups. Let

$$(4.1) \quad L : G\mathcal{T} \longrightarrow H\mathcal{T} \text{ and } R : H\mathcal{T} \longrightarrow G\mathcal{T}$$

be two functors. A *left-to-right homotopical adjunction* is a natural map

$$(4.2) \quad \text{Map}_H(LX, Y) \longrightarrow \text{Map}_G(X, RY),$$

which is a weak equivalence of spaces when X is a G -CW complex.

Analogously, a *right-to-left homotopical adjunction* is a natural map

$$(4.3) \quad \text{Map}_G(X, RY) \longrightarrow \text{Map}_H(LX, Y)$$

which is a weak equivalence of spaces when X is a G -CW complex.

L is called a *homotopical left adjoint* and R a *homotopical right adjoint*.

Let's see an example.

EXAMPLE 4.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ and g be a generator of G . We want to find a homotopical right adjoint R of the functor $X \mapsto X^g$ from the category $G\mathcal{T}$ of G -spaces to the category \mathcal{T} of topological spaces.

Let Y be a topological space. Suppose we have

$$\text{Map}(X^g, Y) \simeq \text{Map}_G(X, RY).$$

G has two subgroups, e and G .

$$RY^e = \text{Map}_G(G/e, RY) \simeq \text{Map}((G/e)^g, Y) \simeq \text{pt};$$

$$RY^G = \text{Map}_G(G/G, RY) \simeq \text{Map}((G/G)^g, Y) = Y.$$

If Y is the empty set, $R\emptyset$ is EG . And generally for any Y , one choice of RY is the join $Y * EG$.

By Elmendorf's theorem 2.3, the space RY is unique up to G -homotopy. By definition, the functor R is a homotopical right adjoint to the fixed point functor $X \mapsto X^g$.

This definition below is Definition 4.5 in Chapter V of [45].

DEFINITION 4.3. A family \mathcal{F} in G is a set of subgroups of G that is closed under subconjugacy: if $H \in \mathcal{F}$ and $g^{-1}Kg \subset H$, then $K \in \mathcal{F}$. An \mathcal{F} -space is a G -space all of whose isotropy groups are in \mathcal{F} . Define a functor $\underline{\mathcal{F}} : h\mathcal{O}_G \rightarrow \text{Sets}$ by sending G/H to the 1-point set if $H \in \mathcal{F}$ and to the empty set if H is not in \mathcal{F} . Define the universal \mathcal{F} -space X of the homotopy type of a G -CW complex, there is one and, up to homotopy, only one G -map $X \rightarrow E\mathcal{F}$. Define the classifying space of the family \mathcal{F} to be the orbit space $B\mathcal{F} = E\mathcal{F}/G$.

For any compact Lie group G , let $\langle g \rangle$ denote the cyclic subgroup of G generated by $g \in G^{\text{tors}}$ and $*$ denote the join. Let

$$S_{G,g} := \text{Map}_{\langle g \rangle}(G, *_K E(\langle g \rangle/K))$$

where K goes over all the maximal subgroups of $\langle g \rangle$ and $E(\langle g \rangle/K)$ is the universal space of the cyclic group $\langle g \rangle/K$. The action of $\langle g \rangle/K$ on $E(\langle g \rangle/K)$ is free.

For this space $S_{G,g}$, it's classified up to G -homotopy, as shown in the following lemma.

LEMMA 4.4. For any closed subgroup $H \leq G$, $S_{G,g}$ satisfies

$$(4.4) \quad S_{G,g}^H \simeq \begin{cases} \text{pt}, & \text{if for any } b \in G, b^{-1}\langle g \rangle b \not\leq H; \\ \emptyset, & \text{if there exists a } b \in G \text{ such that } b^{-1}\langle g \rangle b \leq H. \end{cases}$$

PROOF. For any closed subgroup H of G .

$$(4.5) \quad S_{G,g}^H = \text{Map}_{\langle g \rangle}(G/H, *_K E(\langle g \rangle/K))$$

where K goes over all the cyclic groups $\langle g^m \rangle$ with $\frac{|g|}{m}$ a prime.

If there exists an $b \in G$ such that $b^{-1}\langle g \rangle b \leq H$, it's equivalent to say that there exists points in G/H that can be fixed by g . But there are no points in $*_K E(\langle g \rangle/K)$ that can be fixed by g . So there is no $\langle g \rangle$ -equivariant map from G/H to $*_K E(\langle g \rangle/K)$. In this case $S_{G,g}^H$ is empty.

If for any $b \in G$, $b^{-1}\langle g \rangle b \not\leq H$, it's equivalent to say that there are no points in G/H that can be fixed by g . And for any subgroup $\langle g^m \rangle$ which is not $\langle g \rangle$ itself, $(*_K E(\langle g \rangle/K))^{\langle g^m \rangle}$ is the join of several contractible spaces $E(\langle g \rangle/K)^{\langle g^m \rangle}$, thus, contractible. So all the homotopy groups $\pi_n((*_K E(\langle g \rangle/K))^{\langle g^m \rangle})$ are trivial. For any $n \geq 1$ and any $\langle g \rangle$ -equivariant map

$$f : (G/H)^n \longrightarrow *_K E(\langle g \rangle/K)$$

from the n -skeleton of G/H , the obstruction cocycle is zero.

Then by equivariant obstruction theory, f can be extended to the $(n+1)$ -cells of G/H , and any two extensions f and f' are $\langle g \rangle$ -homotopic.

So in this case $S_{G,g}^H$ is contractible. \square

In Theorem 4.5 we show another example of homotopical right adjoint, which is crucial to the construction of $QEll_G^n$.

THEOREM 4.5. *Let G be a compact Lie group and $g \in G^{tors}$. Consider the functor*

$$L_g : G\mathcal{T} \longrightarrow C_G(g)\mathcal{T}, \quad X \mapsto X^g.$$

A homotopical right adjoint of it is $R_g : C_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with

$$(4.6) \quad R_g Y = \text{Map}_{C_G(g)}(G, Y * S_{C_G(g),g}).$$

PROOF. Let H be any closed subgroup of G .

First I show given a $C_G(g)$ -equivariant map $f : (G/H)^g \longrightarrow Y$, it extends uniquely up to $C_G(g)$ -homotopy to a $C_G(g)$ -equivariant map $\tilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \longrightarrow Y * S_{C_G(g),g}$ by composing with the inclusion of one end of the join

$$Y \longrightarrow Y * S_{C_G(g),g}, \quad y \mapsto (1y, 0).$$

If $bH \in (G/H)^g$, define $\tilde{f}(bH) := f(bH)$.

If bH is not in $(G/H)^g$, its stabilizer group does not contain g . By Lemma 4.4, for any subgroup K of it, $S_{C_G(g),g}^K$ is contractible. So $(Y * S_{C_G(g),g})^K = Y^K * S_{C_G(g),g}^K$ is contractible. In other words, if K occurs as the isotropy subgroup of a point outside $(G/H)^g$, $\pi_n((Y * S_{C_G(g),g})^K)$ is trivial. By equivariant obstruction theory, f can extend to a $C_G(g)$ -equivariant map $\tilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$, and any two extensions are $C_G(g)$ -homotopy equivalent. In addition, $S_{C_G(g),g}^g$ is empty. So the image of the restriction of any map $G/H \longrightarrow Y * S_{C_G(g),g}$ to the subspace $(G/H)^g$ is contained in the end Y of the join.

Thus, $\text{Map}_{C_G(g)}((G/H)^g, Y)$ is weak equivalent to $\text{Map}_{C_G(g)}(G/H, Y * S_{C_G(g),g})$.

Moreover, by the adjunction (2.4) we have the equivalence

$$(4.7) \quad \text{Map}_G\left(G/H, \text{Map}_{C_G(g)}(G, Y * S_{C_G(g),g})\right) \cong \text{Map}_{C_G(g)}(G/H, Y * S_{C_G(g),g})$$

So we get

$$(4.8) \quad R_g Y^H = \text{Map}_G(G/H, R_g Y) \simeq \text{Map}_{C_G(g)}((G/H)^g, Y)$$

Let X be of the homotopy type of a G -CW complex. Let X^k denote the k -skeleton of X . Consider the functors

$$\text{Map}_G(-, R_g Y) \text{ and } \text{Map}_{C_G(g)}((-)^g, Y)$$

from $G\mathcal{T}$ to \mathcal{T} . Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\text{Map}_G(-, R_g Y)$ to $\text{Map}_{C_G(g)}((-)^g, Y)$ by sending a G -map $F : X \rightarrow R_g Y$ to the composition

$$(4.9) \quad X^g \xrightarrow{F^g} (R_g Y)^g \rightarrow Y^g \subseteq Y$$

with the second map $f \mapsto f(e)$. Note that for any $f \in (R_g Y)^g$, $f(e) = (g \cdot f)(e) = f(eg) = f(g) = g \cdot f(e)$ so $f(e) \in (Y * S_{C_G(g),g})^g = Y^g$ and the second map is well-defined. It gives weak equivalence on orbits, as shown in (4.8). Thus, by Proposition 2.4, R_g is a homotopical right adjoint of L . \square

THEOREM 4.6. *Let G be a compact Lie group, $g \in G^{tors}$, and Y a $\Lambda_G(g)$ -space. The subgroup*

$$\{[(1, t)] \in \Lambda_G(g) | t \in \mathbb{R}\}$$

of $\Lambda_G(g)$ is isomorphic to \mathbb{R} . Let's use the same symbol \mathbb{R} to denote it. Consider the functor $\mathcal{L}_g : G\mathcal{T} \rightarrow \Lambda_G(g)\mathcal{T}$, $X \mapsto X^g$ where $\Lambda_G(g)$ acts on X^g by

$$[g, t] \cdot x = gx.$$

The functor $\mathcal{R}_g : \Lambda_G(g)\mathcal{T} \rightarrow G\mathcal{T}$ with

$$(4.10) \quad \mathcal{R}_g Y = \text{Map}_{C_G(g)}(G, Y^{\mathbb{R}} * S_{C_G(g),g})$$

is a homotopical right adjoint of \mathcal{L}_g .

PROOF. Let X be a G -space. Let H be any closed subgroup of G . Note for any G -space X , \mathbb{R} acts trivially on X^g , thus, the image of any $\Lambda_G(g)$ -equivariant map $X^g \rightarrow Y$ is in $Y^{\mathbb{R}}$. So we have

$$\text{Map}_{\Lambda_G(g)}(X^g, Y) = \text{Map}_{C_G(g)}(X^g, Y^{\mathbb{R}}).$$

First I show $f : (G/H)^g \rightarrow Y^{\mathbb{R}}$ extends uniquely up to $C_G(g)$ -homotopy to a $C_G(g)$ -equivariant map $\tilde{f} : G/H \rightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \rightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$ by composing with the inclusion as the end of the join

$$Y^{\mathbb{R}} \rightarrow Y^{\mathbb{R}} * S_{C_G(g),g}, \quad y \mapsto (1y, 0).$$

If $bH \in (G/H)^g$, define $\tilde{f}(bH) = f(bH)$.

If bH is not in $(G/H)^g$, its stabilizer group does not contain g . By Lemma 4.4, for any subgroup K of it, $S_{C_G(g),g}^K$ is contractible. So $(Y^{\mathbb{R}} * S_{C_G(g),g})^K = (Y^{\mathbb{R}})^K * S_{C_G(g),g}^K$ is contractible. In other words, if K occurs as the isotropy subgroup of a point in G/H outside $(G/H)^g$, $\pi_n((Y^{\mathbb{R}} * S_{C_G(g),g})^K)$ is trivial. By equivariant obstruction theory, f can extend to a $C_G(g)$ -equivariant map $\tilde{f} : G/H \rightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$, and any two extensions are $C_G(g)$ -homotopy equivalent. In addition, $S_{C_G(g),g}^g$ is empty. So the image of the restriction of any map $G/H \rightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$ to the subspace $(G/H)^g$ is contained in the end $Y^{\mathbb{R}}$ of the join.

Thus, $\text{Map}_{C_G(g)}((G/H)^g, Y^{\mathbb{R}})$ is weak equivalent to $\text{Map}_{C_G(g)}(G/H, Y^{\mathbb{R}} * S_{C_G(g),g})$.

Moreover, by the adjunction (2.4) we have the equivalence

$$(4.11) \quad \text{Map}_G\left(G/H, \text{Map}_{C_G(g)}(G, Y^{\mathbb{R}} * S_{C_G(g),g})\right) \cong \text{Map}_{C_G(g)}(G/H, Y^{\mathbb{R}} * S_{C_G(g),g})$$

So we get

$$(4.12) \quad \mathcal{R}_g Y^H = \text{Map}_G(G/H, \mathcal{R}_g Y) \simeq \text{Map}_{C_G(g)}((G/H)^g, Y)$$

Let X be a space of the homotopy type of a G -CW complex. Consider the functors

$$\text{Map}_G(-, \mathcal{R}_g Y) \text{ and } \text{Map}_{C_G(g)}((-)^g, Y)$$

from $G\mathcal{T}$ to \mathcal{T} . Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\text{Map}_G(-, \mathcal{R}_g Y)$ to $\text{Map}_{C_G(g)}((-)^g, Y)$ by sending a G -map $F : X \rightarrow \mathcal{R}_g Y$ to the composition

$$(4.13) \quad X^g \xrightarrow{F^g} (\mathcal{R}_g Y)^g \rightarrow Y^g$$

with the second map $f \mapsto f(e)$. Note that for any $f \in (\mathcal{R}_g Y)^g$, $f(e) = (g \cdot f)(e) = f(eg) = f(g) = g \cdot f(e)$ so $f(e) \in (Y^{\mathbb{R}} * S_{C_G(g),g})^g = Y^{\mathbb{R}}$ and the second map is well-defined. It gives weak equivalence on orbits, as shown in (4.12). Thus, by Proposition 2.4, \mathcal{R}_g is a homotopical right adjoint of \mathcal{L}_g . \square

Theorem 4.6 implies Theorem 4.7 directly.

THEOREM 4.7. *For any compact Lie group G and any integer n , Let $KU_{G,n}$ denote the space representing the n -th G -equivariant KU -theory. The n -th quasi-elliptic cohomology*

$$QEll_G^n(X) \cong \prod_{g \in G_{conj}^{tors}} [X^g, KU_{\Lambda_G(g),n}]^{\Lambda_G(g)}$$

is weakly represented by the space

$$QEll_{G,n} := \prod_{g \in G_{conj}^{tors}} \mathcal{R}_g(KU_{\Lambda_G(g),n})$$

in the sense of (1.2) where $\mathcal{R}_g(KU_{\Lambda_G(g),n})$ is the space

$$\text{Map}_{C_G(g)}(G, KU_{\Lambda_G(g),n}^{\mathbb{R}} * S_{C_G(g),g}).$$

5. Equivariant Orthogonal spectra

In Section 5.1, we recall the basics of equivariant orthogonal spectra. There are many references for this topic, such as [11], [42] [57], [44], etc. In Section 5.7 we recall the global K-theory, which is a prominent example of global homotopy theory. Its properties will be applied in the construction of the orthogonal G -spectra for quasi-elliptic cohomology.

5.1. Orthogonal G -spectra. Let G be a compact Lie group. Let \mathcal{I}_G denote the category whose objects are pairs (\mathbb{R}^n, ρ) with ρ a homomorphism from G to $O(n)$ giving \mathbb{R}^n the structure of a G -representation. Morphisms $(\mathbb{R}^m, \mu) \rightarrow (\mathbb{R}^n, \rho)$ are linear isometric isomorphisms $\mathbb{R}^m \rightarrow \mathbb{R}^n$.

Let Top_G denote the category with objects based G -spaces and morphisms continuous based maps.

DEFINITION 5.1. An \mathcal{I}_G -space is a G -continuous functor $X : \mathcal{I}_G \rightarrow Top_G$. Morphisms between \mathcal{I}_G -spaces are natural G -transformations.

EXAMPLE 5.2. The sphere \mathcal{I}_G -space S is the functor $V \mapsto S^V$ which sends a representation to its one-point compactification.

DEFINITION 5.3. An orthogonal G -spectrum is an \mathcal{I}_G -space X together with a natural transformation of functors $\mathcal{I}_G \times \mathcal{I}_G \rightarrow \text{Top}_G$

$$X(-) \wedge S^- \rightarrow X(- \oplus -)$$

satisfying appropriate associativity and unitality diagrams. In other words, an orthogonal G -spectrum is an \mathcal{I}_G -space with an action of the sphere \mathcal{I}_G -space.

DEFINITION 5.4. For \mathcal{I}_G -spaces X and Y , define the "external" smash product $X \overline{\wedge} Y$ by

$$(5.1) \quad X \overline{\wedge} Y = \wedge \circ (X \times Y) : \mathcal{I}_G \times \mathcal{I}_G \rightarrow \text{Top}_G;$$

thus $(X \overline{\wedge} Y)(V, W) = X(V) \wedge Y(W)$.

We have an equivariant notion of a functor with smash product (FSP).

DEFINITION 5.5. An \mathcal{I}_G -FSP is an \mathcal{I}_G -space X with a unit G -map $\eta : S \rightarrow X$ and a natural product G -map $\mu : X \overline{\wedge} X \rightarrow X \circ \bigoplus$ of functors $\mathcal{I}_G \times \mathcal{I}_G \rightarrow \text{Top}_G$ such that the evident unit, associativity and centrality of unit diagram also commutes.

A \mathcal{I}_G -FSP is commutative if the diagram below commutes.

$$\begin{array}{ccc} X(V) \wedge X(W) & \xrightarrow{\mu} & X(V \oplus W) \\ \tau \downarrow & & X(\tau) \downarrow \\ X(W) \wedge X(V) & \xrightarrow{\mu} & X(W \oplus V). \end{array}$$

Note that this diagram commutes implies the centrality of unit diagram commutes.

LEMMA 5.6. An \mathcal{I}_G -FSP has an underlying \mathcal{I}_G -spectrum with structure G -map

$$\sigma = \mu \circ (id \overline{\wedge} \eta) : X \overline{\wedge} S \rightarrow X \circ \bigoplus.$$

5.2. Global K-theory. A classical example of orthogonal spectra is global K-theory. Quasi-elliptic cohomology can be expressed in terms of equivariant K-theory. And this example is especially important for our construction.

In [36] Joachim constructs G -equivariant K-theory as an orthogonal G -spectrum for any compact Lie group G . In fact it is the only known E^∞ -version of equivariant complex K-theory when G is a compact Lie group.

Let G be a compact Lie group. For any real G -representation V , let $\mathcal{C}l_V$ be the Clifford algebra of V and \mathcal{K}_V be the G - C^* -algebra of compact operators on $L^2(V)$. Let $s := C_0(\mathbb{R})$ be the graded G - C^* -algebra of continuous functions on \mathbb{R} vanishing at infinity with trivial G -action. Then the orthogonal G -spectrum for equivariant K-theory defined by Joachim is the lax monoidal functor given by

$$\mathbb{K}_G(V) = \text{Hom}_{C^*}(s, \mathcal{C}l_V \otimes \mathcal{K}_V)$$

of $\mathbb{Z}/2$ -graded $*$ -homomorphisms from s to $\mathcal{C}l_V \otimes \mathcal{K}_V$.

Bohmann showed in her paper [11] that Joachim's model is "global", i.e. the lax monoidal functor \mathbb{K} is an orthogonal \mathcal{G} -spectrum. For more detail, please read [11] for reference.

Schwede's construction of global K-theory KR in [57] is a unitary analog of the construction by Joachim. It is an ultra-commutative ring spectrum whose G -homotopy type realizes Real G -equivariant periodic K-theory.

For any complex inner product space W , let $\Lambda(W)$ be the exterior algebra W and $Sym(W)$ the symmetric algebra of it. The tensor product

$$\Lambda(W) \otimes Sym(W)$$

inherits a hermitian inner product from W and it's $\mathbb{Z}/2$ -graded by even and odd exterior powers. Let \mathcal{H}_W denote the Hilbert space completion of $\Lambda(W) \otimes Sym(W)$. Let \mathcal{K}_W be the C^* -algebra of compact operators on \mathcal{H}_W . The orthogonal spectrum KR is defined to be the lax monoidal functor

$$KR(W) = Hom_{C^*}(s, \mathcal{K}_W).$$

Let uW denote the underlying euclidean vector space of W . There is an isomorphism of $\mathbb{Z}/2$ -graded C^* -algebras

$$Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W)) \cong \mathcal{K}_W.$$

So we get a homeomorphism

$$KR(W) \cong Hom_{C^*}(s, Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W))) = \mathbb{K}(uW).$$

In [57], Schwede shows that the spaces in the orthogonal spectrum KR represent real equivariant K-theory.

THEOREM 5.7. *For an augmented Lie group G , a "large" real G -representation and a compact G -space B , there is a bijection $\Psi_{G,B,V} : K_G(B) \rightarrow [B_+, KR(V)]^G$ that is natural in B . The left hand side is the real G -equivariant K -group of B .*

We have the relations below between the global Real K-theory KR , periodic unitary K-theory KU and periodic orthogonal real K-theory KO .

$$(5.2) \quad KU = u(KR); \quad KO = KR^\psi$$

We will use the orthogonal spectra KU in the construction of orthogonal quasi-elliptic cohomology.

DEFINITION 5.8. An orthogonal G -representation is called ample if its complexified symmetric algebra is complete complex G -universe.

THEOREM 5.9. (i) *Let G be a compact Lie group and V an orthogonal G -representation. For every ample G -representation W , the adjoint structure map*

$$\tilde{\sigma}_{V,W}^K : KU(V) \rightarrow Map(S^W, KU(V \oplus W))$$

is a G -weak equivalence.

(ii) *Let G be an augmented Lie group and V a real G -representation such that $Sym(V)$ is a complete real G -universe. For every real G -representation W the adjoint structure map*

$$\tilde{\sigma}_{V,W}^K : KR(V) \rightarrow Map(S^W, KR(V \oplus W))$$

is a G -weak equivalence.

6. Orthogonal G -spectra of $QEll_G$

In Section 6.3 and 6.4, via the spaces we construct in Section 4, we construct a G -orthogonal spectra for quasi-elliptic cohomology up to weak equivalence (1.2), which is a commutative orthogonal G -spectra. Before that, in Section 6.1 and 6.2, we discuss complex and real $\Lambda_G(\sigma)$ -representations.

6.1. Preliminaries: faithful representations of $\Lambda_G(g)$. Let G be a compact Lie group. As shown in Theorem 5.7, $KU(V)$ represents G -equivariant complex K-theory when V is a faithful G -representation. Therefore, before the construction in Section 6.3, we construct a faithful $\Lambda_G(\sigma)$ -representation from a faithful G -representation.

Let $\sigma \in G^{tors}$ with order l . We construct a functor $(-)_\sigma$ from the category of G -representations to the category of $\Lambda_G(\sigma)$ -representations.

Let ρ be a complex G -representation with underlying space V . Let $i : C_G(\sigma) \hookrightarrow G$ denote the inclusion of groups. The restriction i^*V is a complex $C_G(\sigma)$ -representation.

Let $\{\lambda\}$ denote all the irreducible complex representations of $C_G(\sigma)$. As said in [22], we have the decomposition of a representation into its isotypic components

$$(6.1) \quad i^*V \cong \bigoplus_{\lambda} V_{\lambda}$$

where V_{λ} denotes the sum of all subspaces of V isomorphic to λ . Each

$$V_{\lambda} = Hom_{C_G(\sigma)}(\lambda, V) \otimes_{\mathbb{C}} \lambda$$

is unique as a subspace. Note that σ acts on each V_{λ} as a diagonal matrix.

Let's equip each V_{λ} a $\Lambda_G(\sigma)$ -action, as shown below.

Each $\lambda(\sigma)$ is of the form $e^{\frac{2\pi i m_{\lambda}}{l}} I$ with $0 < m_{\lambda} \leq l$ and I the identity matrix. As shown in Lemma 3.1, $V_{\lambda} \odot_{\mathbb{C}} q^{\frac{m_{\lambda}}{l}}$ is a well-defined complex $\Lambda_G(\sigma)$ -representation. Define

$$(6.2) \quad (V_{\lambda})_{\sigma} := V_{\lambda} \odot_{\mathbb{C}} q^{\frac{m_{\lambda}}{l}}$$

and

$$(6.3) \quad (V)_{\sigma} := \bigoplus_{\lambda} V_{\lambda} \odot_{\mathbb{C}} q^{\frac{m_{\lambda}}{l}}$$

Each $(V_{\lambda})_{\sigma}$ is the isotypic component of $(V)_{\sigma}$ corresponding to the irreducible representation $\lambda \odot_{\mathbb{C}} q^{\frac{m_{\lambda}}{l}}$.

The complex $\Lambda_G(\sigma)$ -representation $(V)_{\sigma}$ has the same dimension as V .

PROPOSITION 6.1. Let V be a faithful G -representation. And let $\sigma \in G^{tors}$.

- (i) If V contains a trivial subrepresentation, $(V)_{\sigma}$ is a faithful $\Lambda_G(\sigma)$ -representation.
- (ii) $(V)_{\sigma} \oplus (V)_{\sigma} \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)$ -representation.
- (iii) $(V)_{\sigma} \oplus V^{\sigma}$ is a faithful $\Lambda_G(\sigma)$ -representation.

PROOF. (i) Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)_{\sigma}$. Assume $t \in [0, 1)$. On $(V_1)_{\sigma}$, $[a, t]v_0 = e^{2\pi i t} v_0 = v_0$. So $t = 0$. Then on the whole space V_{σ} , since $C_G(\sigma)$ acts faithfully on it and for any $v \in V_{\sigma}$, $[a, 0] \cdot v = a \cdot v = v$, then $a = e$.

So $(V)_{\sigma}$ is a faithful $\Lambda_G(\sigma)$ -representation.

(ii) Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on V_{σ} . Consider the subrepresentation $(V_{\lambda})_{\sigma}$ and $(V_{\lambda})_{\sigma} \otimes_{\mathbb{C}} q^{-1}$ of $(V)_{\sigma} \oplus (V)_{\sigma} \otimes_{\mathbb{C}} q^{-1}$ respectively. Let

v be an element in the underlying vector space V_λ . On $(V_\lambda)_\sigma$, $[a, t] \cdot v = e^{\frac{2\pi i m_\lambda t}{t}} a \cdot v = v$; and on $(V_\lambda)_\sigma \otimes_{\mathbb{C}} q^{-1}$, $[a, t] \cdot v = e^{\frac{2\pi i m_\lambda t}{t} - 2\pi i t} a \cdot v = v$. So we get $e^{2\pi i t} \cdot v = v$. Thus, $t = 0$.

$C_G(\sigma)$ acts faithfully on V , so it acts faithfully on $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$. Since $[a, 0] \cdot w = w$, for any $w \in (V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$, so $a = e$.

Thus, $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)$ -representation.

(iii) Note that V^σ with the trivial \mathbb{R} -action is the representation $(V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$. The representation $(V)_\sigma \oplus V^\sigma$ contains a subrepresentation $(V^\sigma)_\sigma \oplus (V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$, which is a faithful $\Lambda_G(\sigma)$ -representation by the second conclusion of Proposition 6.1. So $(V)_\sigma \oplus V^\sigma$ is faithful. \square

LEMMA 6.2. For any $\sigma \in G^{tors}$, $(-)_\sigma$ defined in (6.3) is a functor from the category of G -spaces to the category of $\Lambda_G(\sigma)$ -spaces.

Moreover, $(-)_\sigma \oplus (-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ and $(-)_\sigma \oplus (-)^\sigma$ in Proposition 6.1 are also well-defined functors from the category of G -spaces to the category of $\Lambda_G(\sigma)$ -spaces.

PROOF. Let $f : V \rightarrow W$ be a G -equivariant map. Then f is $C_G(\sigma)$ -equivariant for each $\sigma \in G^{tors}$. For each irreducible complex $C_G(\sigma)$ -representation λ , $f : V_\lambda \rightarrow W_\lambda$ is $C_G(\sigma)$ -equivariant. And

$$f_\sigma : (V_\lambda)_\sigma \rightarrow (W_\lambda)_\sigma, v \mapsto f(v)$$

with the same underlying spaces is well-defined and is $\Lambda_G(\sigma)$ -equivariant.

It is straightforward to check if we have two G -equivariant maps $f : V \rightarrow W$ and $g : U \rightarrow V$, then

$$(f \circ g)_\sigma = f_\sigma \circ g_\sigma.$$

So $(-)_\sigma$ gives a well-defined functor from the category of G -representations to the category of $\Lambda_G(\sigma)$ -representation.

Similarly, we can see $(-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is also a well-defined functor from the category of G -representations to the category of $\Lambda_G(\sigma)$ -representation, so $(-)_\sigma \oplus (-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is.

Since the fixed point functor $(-)^\sigma$ is also a functor from the category of G -spaces to the category of $\Lambda_G(\sigma)$ -spaces, $(-)_\sigma \oplus (-)^\sigma$ is as well. \square

PROPOSITION 6.3. Let H and G be two compact Lie groups. Let $\sigma \in G$ and $\tau \in H$. Let V be a G -representation and W a H -representation.

(i) We have the isomorphisms of representations below.

$$(V \oplus W)_{(\sigma, \tau)} = (V_\sigma \oplus W_\tau)$$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ -representations.

$$(V \oplus W)_{(\sigma, \tau)} \oplus (V \oplus W)_{(\sigma, \tau)} \otimes_{\mathbb{C}} q^{-1} = ((V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}) \oplus ((W)_\tau \oplus (W)_\tau \otimes_{\mathbb{C}} q^{-1})$$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ -representations.

And

$$(V \oplus W)_{(\sigma, \tau)} \oplus (V \oplus W)^{(\sigma, \tau)} = ((V)_\sigma \oplus V^\sigma) \oplus ((W)_\tau \oplus W^\tau)$$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ —representations.

(ii) Let $\phi : H \rightarrow G$ be a group homomorphism. Let $\phi_\tau : \Lambda_H(\tau) \rightarrow \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from ϕ . Then we have

$$\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau,$$

$$\phi_\tau^*((V)_{\phi(\tau)} \oplus (V)_{\phi(\tau)} \otimes_{\mathbb{C}} q^{-1}) = (V)_\tau \oplus (V)_\tau \otimes_{\mathbb{C}} q^{-1},$$

and

$$\phi_\tau^*((V)_{\phi(\tau)} \oplus V^{\phi(\tau)}) = (V)_\tau \oplus V^\tau$$

as $\Lambda_H(\tau)$ —representations.

PROOF. (i) Let

$$\{\lambda_G\} \text{ and } \{\lambda_H\}$$

denote the sets of all the irreducible $C_G(\sigma)$ —representations and all the irreducible $C_H(\tau)$ —representations.

λ_G and λ_H are irreducible representations of $C_{G \times H}(\sigma, \tau)$ via the inclusion $C_G(\sigma) \rightarrow C_{G \times H}(\sigma, \tau)$ and $C_H(\tau) \rightarrow C_{G \times H}(\sigma, \tau)$.

The \mathbb{R} —representation assigned to each $C_{G \times H}(\sigma, \tau)$ —irreducible representation in $V \oplus W$ is the same as that assigned to the irreducible representations of V and W .

So we have

$$(V \oplus W)_{(\sigma, \tau)} = (V)_\sigma \oplus (W)_\tau$$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ —representations.

Similarly we can prove the other two conclusions in (i).

(ii) Let $\sigma = \phi(\tau)$. If $(\phi_\tau^* V)_{\lambda_H}$ is a $C_H(\tau)$ —subrepresentation of $\phi_\tau^* V_{\lambda_G}$, the \mathbb{R} —representation assigned to it is the same as that to V_{λ_G} .

So we have

$$\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau$$

as $\Lambda_H(\tau)$ —representations.

Similarly we can prove the other two conclusions in (ii). □

6.2. Real $\Lambda_G(\sigma)$ —representation. In this section we discuss real $\Lambda_G(\sigma)$ —representation and its relation with the complex $\Lambda_G(\sigma)$ —representations introduced in Lemma 3.1.

Let G be a compact Lie group, $\sigma \in G^{tors}$. For real representations of $\Lambda_G(\sigma)$, the case is a little complicated. First let's recall some definitions and conclusions in real representation theory. The main reference is [13] and [22].

DEFINITION 6.4. A complex representation $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is said to be self dual if it is isomorphic to its complex dual $\rho^* : G \rightarrow \text{Aut}_{\mathbb{C}}(V^*)$ where $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\rho^*(g) = \rho(g^{-1})^*$.

An irreducible complex representation $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is said to be of **real type** if satisfies the equivalent conditions (1-3).

(1) $V = \mathbb{C} \otimes U$ is the complexification of a real representation $G \rightarrow \text{Aut}_{\mathbb{R}}(U)$.

(2) V admits an equivariant real structure, i.e. an antilinear map $S : V \rightarrow V$ such that $S^2(v) = v$.

- (3) There is an equivariant isomorphism $B : V \longrightarrow V^*$ such that $B^* = B$.

An irreducible complex representation is said to be of **quaternionic type** if it satisfies the equivalent conditions (4-6).

- (4) $V = W_{\mathbb{C}}$ is obtained from a quaternionic representation $G \longrightarrow \text{Aut}_{\mathbb{H}}(W)$ by restriction of scalars $\mathbb{C} \subset \mathbb{H}$.

- (5) V admits an equivariant quaternionic structure, i.e. an antilinear map $S : V \longrightarrow V$ such that $S^2(v) = -v$.

- (6) There is an equivariant isomorphism $B : V \longrightarrow V^*$ such that $B^* = -B$.

An irreducible complex representation is said to be of **complex type** if it's not self dual.

DEFINITION 6.5. A complex representation is said to have an **irreducible real form** if it is the complexification of an irreducible real representation.

LEMMA 6.6. *An irreducible complex representation V is of*

- (I) *real type if and only if V has irreducible real form.*
- (II) *complex type if and only if $V \oplus V^*$ has irreducible real form.*
- (III) *quaternionic type if and only if $V \oplus V$ has irreducible real form.*

EXAMPLE 6.7 (real representation ring of circle). We know the complex representation ring $R\mathbb{T}$ of a circle is $\mathbb{Z}[q, q^{-1}]$ where $q : \mathbb{T} \longrightarrow U(1)$ is the isomorphism class of irreducible complex representation sending $e^{2\pi it}$ to $(e^{2\pi it})$. The real representation ring $RO(\mathbb{T})$ is the subring of $R\mathbb{T}$ fixed by the involution on it given by $q \mapsto q^{-1}$.

Let $f(q)$ be any polynomial in q and let $f(q) = f_+(q) + f_-(q) + n \cdot 1$ where $f_+(q)$ is the part in $f(q)$ with positive power in q , $f_-(q)$ is the part with negative power and 1 is the trivial representation. $f(q)$ represents an element in $RO(\mathbb{T})$ if and only if f_+ and f_- are the same polynomial. Let V be the representation space of f_+ . Then V^* is the representation space of f_- . So a real representation of \mathbb{T} is always of the form

$$V \oplus V^* \oplus n\mathbb{R}$$

for some complex representation V of \mathbb{T} and nonnegative integer n .

EXAMPLE 6.8. Let $\rho : C_G(g) \longrightarrow \text{Aut}_{\mathbb{R}}(V)$ be an irreducible complex $C_G(g)$ -representation. Then as in Lemma 3.1, there exists a character $\eta : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\rho(g) = \eta(1)I$. And $\rho \odot_{\mathbb{C}} \eta$ is an irreducible complex representation of $\Lambda_G(g)$. Since $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t)$, it's not self-dual if η is nontrivial. In this case it's of complex type. By Lemma 6.6, $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ has irreducible real form.

If V is of real type, it is the complexification of a real $C_G(g)$ -representation W . If $g = e$ and the character η we choose is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is self-dual. In this case W is a real $\Lambda_G(g)$ -representation via $[\alpha, t] \cdot w = \alpha w$. And $V \odot_{\mathbb{C}} \eta$ is of real type since it's the complexification of W . For any nontrivial element g in G^{tors} , the $\Lambda_G(g)$ -representation $V \odot_{\mathbb{C}} \eta$ is of complex type, then $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ is of the real type.

If V is of quaternion type, then $V = U_{\mathbb{C}}$ can be obtained from a quaternion $C_G(g)$ -representation U by restricting the scalar to \mathbb{C} . If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)$

$\eta)[\alpha, t]$ since V is self-dual. In this case W is a quaternion $\Lambda_G(g)$ –representation with $[\alpha, t] \cdot w = \alpha w$. So $V \odot_{\mathbb{C}} \eta$ is of quaternion type.

Consider the case that V is of complex type. If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) \neq (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is not self-dual. So $V \odot_{\mathbb{C}} \eta$ is of complex type.

For any compact Lie group, let's use $RO(G)$ denote the real representation ring of G . In light of the analysis in Example 6.8, we have the following conclusion.

LEMMA 6.9. *Let $\sigma \in G^{tors}$. Then the map $\pi^* : RO\mathbb{T} \rightarrow RO\Lambda_G(\sigma)$ exhibits $RO\Lambda_G(\sigma)$ as a free $RO\mathbb{T}$ –module.*

In particular there is an $RO\mathbb{T}$ –basis of $RO\Lambda_G(\sigma)$ given by irreducible real representations $\{V_\Lambda\}$. There is a bijection between $\{V_\Lambda\}$ and the set $\{\lambda\}$ of irreducible real representations of $C_G(\sigma)$. When σ is trivial, V_Λ has the same underlying space V as λ . When σ is nontrivial, $V_\Lambda = ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta) \oplus ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta)^$ where η is a complex \mathbb{R} –representation such that $(\lambda \otimes_{\mathbb{R}} \mathbb{C})(\sigma)$ acts on $V \otimes_{\mathbb{R}} \mathbb{C}$ via the scalar multiplication by $\eta(1)$. The dimension of V_Λ is twice as that of λ .*

As in (6.3), we can construct a functor $(-)_\sigma^{\mathbb{R}}$ from the category of real G –representations to the category of real $\Lambda_G(\sigma)$ –representations with

$$(6.4) \quad (V)_\sigma^{\mathbb{R}} = (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma \oplus (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma^*.$$

Similarly we have the conclusion below.

PROPOSITION 6.10. *Let V be a faithful real G –representation. And let $\sigma \in G^{tors}$ and l denote its order. Then $(V)_\sigma^{\mathbb{R}}$ is a faithful real $\Lambda_G(\sigma)$ –representation.*

PROOF. Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)_\sigma^{\mathbb{R}}$. Assume $t \in [0, 1)$. Let $v \in (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma$ and let v^* denote its correspondence in $(V \otimes_{\mathbb{R}} \mathbb{C})_\sigma^*$. Then $[a, t] \cdot (v + v^*) = (ae^{2\pi imt} + ae^{-2\pi imt})(v + v^*) = v + v^*$ where $0 < m \leq l$ is determined by σ . Thus a is equal to both $e^{2\pi imt}I$, and $e^{-2\pi imt}I$. Thus $t = 0$ and a is trivial.

So $(V)_\sigma^{\mathbb{R}}$ is a faithful real $\Lambda_G(\sigma)$ –representation. □

Moreover, we have

PROPOSITION 6.11. *Let H and G be two compact Lie groups. Let $\sigma \in G$ and $\tau \in H$. Let V be a real G –representation and W a real H –representation.*

(i) *We have the isomorphisms of representations below.*

$$(V \oplus W)_{(\sigma, \tau)}^{\mathbb{R}} = (V_\sigma^{\mathbb{R}} \oplus W_\tau^{\mathbb{R}})$$

as $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ –representations.

(ii) *Let $\phi : H \rightarrow G$ be a group homomorphism. Let $\phi_\tau : \Lambda_H(\tau) \rightarrow \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from ϕ . Then we have*

$$\phi_\tau^*(V)_{\phi(\tau)}^{\mathbb{R}} = (V)_\tau^{\mathbb{R}},$$

as $\Lambda_H(\tau)$ –representations.

The proof is left to the readers.

6.3. The Construction of $E(G, V)$. Let G be any compact Lie group. In this section we show that there is a \mathcal{I}_G -space $E(G, -)$ such that for each faithful real G -representation V , $E(G, V)$ weakly represents $QEll_G^V(-)$.

First we need an orthogonal version of the space $S_{G,g}$ satisfying the condition in Lemma 4.4.

Let $g \in G^{tors}$ and V a real G -representation. Let $Sym^n(V)$ denote the n -th symmetric power $V^{\otimes n}$, which has an evident $G \wr \Sigma_n$ -action on it. And let

$$Sym(V) := \bigoplus_{n \geq 0} Sym^n(V).$$

If V is an ample G -representation, $Sym(V)$ is a G -representation containing all the irreducible G -representations. Since V is faithful G -representation, for any closed subgroup H of G , $Sym(V)$ is a faithful H -representation and, thus, complete H -universe.

Let $S(G, V)_g$ be the space

$$(6.5) \quad Sym(V) \setminus Sym(V)^g.$$

It has an involution induced by the complex conjugation on V . For any subgroup H of G containing g , $S(H, V)_g$ has the same underlying space as $S(G, V)_g$.

PROPOSITION 6.12. Let V be an orthogonal G -representation. For any closed subgroup $H \leq C_G(g)$, $S(G, V)_g$ satisfies

$$(6.6) \quad S(G, V)_g^H \simeq \begin{cases} \text{pt}, & \text{if } \langle g \rangle \not\leq H; \\ \emptyset, & \text{if } \langle g \rangle \leq H. \end{cases}$$

PROOF. If $\langle g \rangle \leq H$, $Sym(V)^H$ is a subspace of $Sym(V)^g$, so $(Sym(V) \setminus Sym(V)^g)^H$ is empty.

If $\langle g \rangle \not\leq H$, g is not in H . To simplify the symbol, let $Sym^{n,\perp}$ denote the orthogonal complement of $Sym^n(V)^g$ in $Sym^n(V)$.

$$\begin{aligned} (Sym(V) \setminus Sym(V)^g)^H &= \text{colim}_{n \rightarrow \infty} Sym^n(V)^H \setminus (Sym^n(V)^g)^H \\ &= \text{colim}_{n \rightarrow \infty} (Sym^n(V)^g)^H \times ((Sym^{n,\perp})^H \setminus \{0\}) \end{aligned}$$

Let k_n denote the dimension of $(Sym^{n,\perp})^H$. Then

$$(6.7) \quad (Sym^{n,\perp})^H \setminus \{0\} \simeq S^{k_n-1}.$$

As n goes to infinity, k_n goes to infinity. When k_n is large enough, S^{k_n-1} is contractible. So

$$(Sym(V) \setminus Sym(V)^g)^H$$

is contractible.

So we have proved the conclusion. \square

If V is a faithful G -representation, by Proposition 6.10, $(V)_g^{\mathbb{R}}$ is a faithful $\Lambda_G(g)$ -representation. And we consider V^g as a $\Lambda_G(g)$ -representation with trivial \mathbb{R} -action. Then by Theorem 5.7, $KU((V)_g^{\mathbb{R}} \oplus V^g)$ represents $K_{\Lambda_G(g)}^{(V)_g^{\mathbb{R}} \oplus V^g}(-)$. $(V)_g^{\mathbb{R}}$ is not always an ample orthogonal $\Lambda_G(g)$ -representation, thus,

$$\text{Map}(S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g))$$

is not $\Lambda_G(g)$ –weak equivalent to $KU(V^g)$. But it does represent $K_{\Lambda_G(g)}^{V^g}(-)$, as shown below.

Let X be a G –space.

$$\begin{aligned} [X^g, \text{Map}(S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g))]^{\Lambda_G(g)} &= [X^g \wedge S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g)]^{\Lambda_G(g)} \\ &= K_{\Lambda_G(g)}^{(V)_g^{\mathbb{R}} \oplus V^g}(X^g \wedge S^{(V)_g^{\mathbb{R}}}) = K_{\Lambda_G(g)}^{V^g}(X^g). \end{aligned}$$

To simplify the symbol, let's use

$$F_g(G, V)$$

to denote the space $\text{Map}_{\mathbb{R}}(S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g))$. The basepoint c_0 of it is the constant map from $S^{(V)_g^{\mathbb{R}}}$ to the basepoint of $KU((V)_g^{\mathbb{R}} \oplus V^g)$.

For $F_g(G, V)$, we have the conclusions below.

PROPOSITION 6.13. Let G and H be compact Lie groups. Let V be a real G –representation and W a real H –representation. Let $g \in G^{\text{tors}}$, $h \in H^{\text{tors}}$.

(i) $F_g : (G, V) \mapsto F_g(G, V)$ is a functor from \mathcal{I}_G to the category $C_G(g)\mathcal{T}$ of $C_G(g)$ –spaces.

(ii) We have the unit map

$$\eta_g(G, V) : S^{V^g} \longrightarrow F_g(G, V)$$

and the multiplication

$$\mu_{(g,h)}^F((G, V), (H, W)) : F_g(G, V) \wedge F_h(H, W) \longrightarrow F_{(g,h)}(G \times H, V \oplus W)$$

making the unit, associativity and centrality of unit diagram commute.

And $\eta_g(G, V)$ is $C_G(g)$ –equivariant and $\mu_{(g,h)}^F((G, V), (H, W))$ is $C_{G \times H}(g, h)$ –equivariant.

(iii) Let Δ_G denote the diagonal map

$$G \longrightarrow G \times G, \quad g \mapsto (g, g).$$

Let $\tilde{\sigma}_g(G, V, W) : F_g(G, V) \longrightarrow \text{Map}(S^{W^g}, F_g(G, V \oplus W))$ denote the map

$$x \mapsto \left(w \mapsto (\Delta_G^* \circ \mu_{(g,h)}^F)((G, V), (G, W))(x, \eta_g(G, W)(w)) \right).$$

Then $\tilde{\sigma}_g(G, V, W)$ is a $\Lambda_G(g)$ –weak equivalence when V is an ample G –representation.

(iv) We have

$$(6.8) \quad \mu_{(g,h)}^F((G, V), (H, W))(x \wedge y) = \mu_{(h,g)}^F((H, V), (G, W))(y \wedge x)$$

for any $x \in F_g(G, V)$ and $y \in F_h(H, W)$.

PROOF. (i) Let V_1 and V_2 be orthogonal G –representations and $f : V_1 \longrightarrow V_2$ be a linear isometric isomorphism. f gives the linear isometric isomorphisms $f_1 : (V_1)_g^{\mathbb{R}} \longrightarrow (V_2)_g^{\mathbb{R}}$, and $f_2 : (V_1)_g^{\mathbb{R}} \oplus V_1^g \longrightarrow (V_2)_g^{\mathbb{R}} \oplus V_2^g$. Then define $F_g(f) : F_g(V_1) \longrightarrow F_g(V_2)$ in this way: for any \mathbb{R} –equivariant map $\alpha : S^{(V_1)_g^{\mathbb{R}}} \longrightarrow KU((V_1)_g^{\mathbb{R}} \oplus V_1^g)$, $F_g(f)(\alpha)$ is the composition

$$(6.9) \quad S^{(V_2)_g^{\mathbb{R}}} \xrightarrow{S(f_1^{-1})} S^{(V_1)_g^{\mathbb{R}}} \xrightarrow{\alpha} KU((V_1)_g^{\mathbb{R}} \oplus V_1^g) \xrightarrow{KU(f_2)} KU((V_2)_g^{\mathbb{R}} \oplus V_2^g)$$

which is still \mathbb{R} –equivariant.

It's straightforward to check $F_g(\text{Id})$ is the identity map, and for morphisms

$$V_1 \xrightarrow{f} V_2 \xrightarrow{f'} V_3 \text{ in } \mathcal{I}_G, \text{ we have } F_g(f' \circ f) = F_g(f') \circ F_g(f).$$

So we have a well-defined functor $F_g : \mathcal{I}_G \longrightarrow C_G(g)\mathcal{T}$.

(ii) Define the unit map $\eta_g(G, V) : S^{V^g} \longrightarrow F_g(G, V)$ by

$$(6.10) \quad v \mapsto (v' \mapsto \eta_{(V)_g^{\mathbb{R}} \oplus V^g}^K(v \wedge v'))$$

where $\eta_{(V)_g^{\mathbb{R}} \oplus V^g}^K : S^{(V)_g^{\mathbb{R}} \oplus V^g} \longrightarrow KU((V)_g^{\mathbb{R}} \oplus V^g)$ is the unit map for global K-theory. Since $(V)_g^{\mathbb{R}} \oplus V^g$ is a $\Lambda_G(g)$ -representation, $\eta_{(V)_g^{\mathbb{R}} \oplus V^g}^K$ is $\Lambda_G(g)$ -equivariant. So $\eta_g(G, V)$ is well-defined and $\Lambda_G(g)$ -equivariant.

Define the multiplication $\mu_{(g,h)}^F((G, V), (H, W)) : F_g(G, V) \wedge F_h(H, W) \longrightarrow F_{(g,h)}(G \times H, V \oplus W)$ by

$$(6.11) \quad \alpha \wedge \beta \mapsto (v \wedge w \mapsto \mu_{V,W}^K(\alpha(v) \wedge \beta(w)))$$

where $\mu_{V,W}^K$ is the multiplication for global K-theory.

Since $\mu_{V,W}^K$ is $\Lambda_G(g) \times \Lambda_H(h)$ -equivariant, $\mu_{(g,h)}^F((G, V), (H, W))$ is $C_{G \times H}(g, h)$ -equivariant.

It's straightforward to check the unit map and multiplication make the unit, associativity and centrality of unit diagram commute.

(iii) Since V is a faithful G -representation, by Proposition 6.1, $(V)_g^{\mathbb{R}} \oplus V^g$ is a faithful $\Lambda_G(g)$ -representation. By Theorem 5.9, we have the $\Lambda_G(g)$ -weak equivalence

$$KU((V)_g^{\mathbb{R}} \oplus V^g) \xrightarrow{\tilde{\sigma}^K} \text{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))$$

where $\tilde{\sigma}^K$ is the right adjoint of the structure map of the global complex K-theory KU . Thus we have the $\Lambda_G(g)$ -weak equivalence

$$\begin{aligned} \text{Map}(S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g)) &\longrightarrow \text{Map}(S^{(V)_g^{\mathbb{R}}}, \text{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))) \\ &= \text{Map}(S^{W^g}, \text{Map}(S^{(V \oplus W)_g^{\mathbb{R}}}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))), \end{aligned}$$

i.e. $F_g(G, V) \simeq_{C_G(g)} \text{Map}(S^{W^g}, F_g(G, V \oplus W))$.

(iv) (6.8) comes directly from the commutativity of the orthogonal spectrum KU . \square

In Theorem 4.7 I construct a G -space $QEll_G$ representing $QEll_G^*(-)$. With $F_g(G, V)$ and $S(G, V)_g$ just constructed, we can go further than that. Apply Theorem 4.6, we get the conclusion below.

PROPOSITION 6.14. Let V be a faithful orthogonal G -representation. Let $B'(G, V)$ denote the space

$$\prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, F_g(G, V) * S(G, V)_g).$$

$QEll_G^V(-)$ is weakly represented by $B'(G, V)$ in the sense

$$(6.12) \quad \pi_0(B'(G, V)) = QEll_G^V(S^0).$$

The proof of Proposition 6.14 is analogous to that of Theorem 4.7 step by step. Below is the main theorem in Section 6.3. Let's use formal linear combination

$$t_1 a + t_2 b \text{ with } 0 \leq t_1, t_2 \leq 1, t_1 + t_2 = 1$$

to denote points in join, as talked in Appendix A.

PROPOSITION 6.15. Let $E_g(G, V)$ denote

$$\{t_1a + t_2b \in F_g(G, V) * S(G, V)_g \mid \|b\| \leq t_2\} / \{t_1c_0 + t_2b\}.$$

It is the quotient space of a closed subspace of the join $F_g(G, V) * S(G, V)_g$ with all the points of the form $t_1c_0 + t_2b$ collapsed to one point, which I pick as the basepoint of $E_g(G, V)$, where c_0 is the basepoint of $F_g(G, V)$. $E_g(G, V)$ has the evident $C_G(g)$ -action. And it is $C_G(g)$ -weak equivalent to $F_g(G, V) * S(G, V)_g$. As a result, $\prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V))$ is G -weak equivalent to $\prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, F_g(G, V) * S(G, V)_g)$. So when V is a faithful G -representation,

$$(6.13) \quad E(G, V) := \prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V))$$

weakly represents $QEll_G^V(-)$ in the sense

$$(6.14) \quad \pi_0(E(G, V)) \cong QEll_G^V(S^0).$$

PROOF. First I show $F_g(G, V) * S(G, V)_g$ is $C_G(g)$ -homotopy equivalent to

$$E'_g(G, V) := \{t_1a + t_2b \in F_g(G, V) * S(G, V)_g \mid \|b\| \leq t_2\}.$$

Note that $b \in S(G, V)_g$ is never zero. Let $j : E'_g(G, V) \rightarrow F_g(G, V) * S(G, V)_g$ be the inclusion. Let $p : F_g(G, V) * S(G, V)_g \rightarrow E'_g(G, V)$ be the $C_G(g)$ -map sending $t_1a + t_2b$ to $t_1a + t_2 \frac{\min\{\|b\|, t_2\}}{\|b\|} b$. Both j and p are both continuous and $C_G(g)$ -equivariant. $p \circ j$ is the identity map of $E'_g(G, V)$. We can define a $C_G(g)$ -homotopy

$$H : (F_g(G, V) * S(G, V)_g) \times I \rightarrow F_g(G, V) * S(G, V)_g$$

from the identity map on $F_g(G, V) * S(G, V)_g$ to $j \circ p$ by shrinking. For any $t_1a + t_2b \in F_g(G, V) * S(G, V)_g$, Define

$$(6.15) \quad H(t_1a + t_2b, t) := t_1a + t_2((1-t)b + t \frac{\min\{\|b\|, t_2\}}{\|b\|} b).$$

Then I show $E'_g(G, V)$ is G -weak equivalent to $E_g(G, V)$.

Let $q : E'_g(G, V) \rightarrow E_g(G, V)$ be the quotient map. Let H be a closed subgroup of $C_G(g)$.

If g is in H , since $S(G, V)_g^H$ is empty, so $E_g(G, V)^H$ is in the end $F_g(G, V)$ and can be identified with $F_g(G, V)^H$. In this case q^H is the identity map.

If g is not in H , $E'_g(G, V)^H$ is contractible. The cone $\{c_0\} * S(G, V)_g^H$ is contractible, so $q(\{c_0\} * S(G, V)_g^H) = q(\{c_0\} * S(G, V)_g^H)$ is contractible. Note that the subspace of all the points of the form $t_1c_0 + t_2b$ for any t_1 and b is $q(\{c_0\} * S(G, V)_g^H)$. Therefore, $E_g(G, V)^H = E'_g(G, V)^H / q(\{c_0\} * S(G, V)_g^H)$ is contractible.

Therefore, $E'_g(G, V)$ is G -weak equivalent to $F_g(G, V) * S(G, V)_g$. □

In fact, for any based $C_G(g)$ -space Y , we have the general conclusion below.

PROPOSITION 6.16. Let $g \in G^{tors}$. Let Y be a based $\Lambda_G(g)$ -space. Let \tilde{Y}_g denote the $C_G(g)$ -space

$$\{t_1a + t_2b \in Y^{\mathbb{R}} * S(G, V)_g \mid \|b\| \leq t_2\} / \{t_1y_0 + t_2b\}.$$

It is the quotient space of a closed subspace of $Y^{\mathbb{R}} * S(G, V)_g$ with all the points of the form $t_1y_0 + t_2b$ collapsed to one point, i.e the basepoint of \tilde{Y}_g , where y_0 is the basepoint of Y . \tilde{Y}_g is $C_G(g)$ -weak equivalent to $Y^{\mathbb{R}} * S(G, V)_g$. As a result, the functor $R_g : C_G(g)\mathcal{T} \rightarrow G\mathcal{T}$ with

$$(6.16) \quad R_g \tilde{Y} = \text{Map}_{C_G(g)}(G, \tilde{Y}_g)$$

is a homotopical right adjoint of $L : G\mathcal{T} \rightarrow C_G(g)\mathcal{T}$, $X \mapsto X^g$.

The proof of Proposition 6.16 is analogous to that of Theorem 4.6 and Proposition 6.15.

REMARK 6.17. We can consider $E_g(G, V)$ as a quotient space of a subspace of $F_g(G, V) \times \text{Sym}(V) \times I$

$$(6.17) \quad \{(a, b, t) \in F_g(G, V) \times \text{Sym}(V) \times I \mid \|b\| \leq t; \text{ and } b \in S(G, V)_g \text{ if } t \neq 0\}$$

by identifying points $(a, b, 1)$ with $(a', b, 1)$, and collapsing all the points (c_0, b, t) for any b and t . In other words, the end $F_g(G, V)$ in the join $F_g(G, V) * S(G, V)_g$ is identified with the points of the form $(a, 0, 0)$ in (6.17).

In Section 6.4 we need this identification of $E_g(G, V)$ in mind to prove the structure maps are well-defined and continuous.

PROPOSITION 6.18. For each $g \in G^{tors}$,

$$E_g : \mathcal{I}_G \rightarrow C_G(g)\mathcal{T}, (G, V) \mapsto E_g(G, V)$$

is a well-defined functor. As a result,

$$E : \mathcal{I}_G \rightarrow G\mathcal{T}, (G, V) \mapsto \prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V))$$

is a well-defined functor.

PROOF. Let V and W be G -representations and $f : V \rightarrow W$ a linear isometric isomorphism. Then f induces a $C_G(g)$ -homeomorphism $F_g(f)$ from $F_g(G, V)$ to $F_g(G, W)$ and a $C_G(g)$ -homeomorphism $S_g(f)$ from $S(G, V)_g$ to $S(G, W)_g$. We have the well-defined map

$$E_g(f) : E_g(G, V) \rightarrow E_g(G, W)$$

sending a point represented by $t_1a + t_2b$ in the join to that represented by $t_1F_g(f)(a) + t_2S_g(f)(b)$.

It's straightforward to check $E_g(Id)$ is the identity map and the composition law holds.

$E(f) : E(G, V) \rightarrow E(G, W)$ is defined by

$$\prod_{g \in G_{conj}^{tors}} \alpha_g \mapsto \prod_{g \in G_{conj}^{tors}} E_g(f) \circ \alpha_g.$$

It's straightforward to check that it's well-defined, $E(Id)$ is identity and the composition law holds. \square

6.4. Structure Maps. In this section we construct a unit map η^E and a multiplication μ^E so that we get an orthogonal G -spectrum and \mathcal{I}_G -FSP that represents quasi-elliptic cohomology.

Let G and H be compact Lie groups, V an orthogonal G -representation and W an orthogonal H -representation. Let's use x_g to denote the basepoint of $E_g(G, V)$, which is defined in Proposition 6.15.

Let $g \in G^{tors}$. For each $v \in S^V$, there are $v_1 \in S^{V^g}$ and $v_2 \in S^{(V^g)^\perp}$ such that $v = v_1 \wedge v_2$.

Let $\eta_g^E(G, V) : S^V \longrightarrow E_g(G, V)$ be the map

$$(6.18) \quad \eta_g^E(G, V)(v) := \begin{cases} (1 - \|v_2\|)\eta_g(G, V)(v_1) + \|v_2\|v_2, & \text{if } \|v_2\| \leq 1; \\ x_g, & \text{if } \|v_2\| \geq 1. \end{cases}$$

where $\eta_g(G, V)$ is the unit map defined in Proposition 6.13.

LEMMA 6.19. *The map $\eta_g^E(G, V)$ defined in (6.18) is well-defined, continuous and $C_G(g)$ -equivariant.*

PROOF. When v_1 is infinity, $\eta_g(G, V)(v_1)$ is the basepoint of $F_g(V)$. So by the construction of $E_g(G, V)$ in Proposition 6.15, $v = v_1 \wedge v_2$ is mapped to the basepoint of $E_g(G, V)$.

When v_2 is infinity, $\eta_g^E(G, V)(v)$ is the basepoint by definition. So $\eta_g^E(G, V)$ is well-defined. And since $\eta_g(G, V)$ is $C_G(g)$ -equivariant, $\eta_g^E(G, V)$ is $C_G(g)$ -equivariant.

Next I prove $\eta_g^E(G, V)$ is continuous by showing for each point in $E_g(G, V)$, there is an open neighborhood of it whose preimage is open in S^V .

Consider a point x in the image of $\eta_g^E(G, V)$ represented by $t_1 a + t_2 b$.

Case I: $0 < t_2 < 1$ and a is not the basepoint of $F_g(G, V)$.

Let A be an open neighborhood of a in $F_g(G, V)$ not including the basepoint. We can find such an A since $F_g(G, V)$ is Hausdorff. Let $\delta > 0$ be a small enough value. Let $U_{x,\delta}$ be the open neighborhood of x

$$U_{x,\delta} := \{[s_1\alpha + s_2\beta] \in E_g(G, V) \mid \alpha \in A, |s_2 - t_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta_g^E(G, V)^{-1}(U_{x,\delta})$ is the smash product of $\eta_g(G, V)^{-1}(A)$, which is open in S^{V^g} , and an open subset of $S^{(V^g)^\perp}$

$$\{w \in S^{(V^g)^\perp} \mid t_2 - \delta < \|w\| < t_2 + \delta, \|w - b\| < \delta\}.$$

So it's open in S^V .

Case II: $t_2 = 0$ and a is not the basepoint of $F_g(G, V)$.

Let A be an open neighborhood of a in $F_g(G, V)$ not including the basepoint. Let $\delta > 0$ be a small enough value. Let $W_{x,\delta}$ be the open neighborhood of x

$$W_{x,\delta} := \{[s_1\alpha + s_2\beta] \in E_g(G, V) \mid \alpha \in A, |s_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta_g^E(G, V)^{-1}(W_{x,\delta})$ is the smash product of $\eta_g(G, V)^{-1}(A)$, which is open in S^{V^g} , and an open subset of $S^{(V^g)^\perp}$

$$\{w \in S^{(V^g)^\perp} \mid \|w\| < \delta, \|w - b\| < \delta\}.$$

So it's open in S^V .

Case III: x is the basepoint x_g of $E_g(G, V)$.

Let A_0 be an open neighborhood of the basepoint c_0 .

For any point w of the form $t_1 c_0 + t_2 b$ in the space $E'_g(G, V)$ with $0 < t_2 < 1$, let U_{w, δ_w} denote the open subset of $E_g(G, V)$

$$\{[s_1 \alpha + s_2 \beta] \in E_g(G, V) \mid \alpha \in A_0, |s_2 - t_2| < \delta_w, \|\beta - b\| < \delta_w\}$$

with δ_w small enough.

Let W_δ denote the open subset of $E_g(G, V)$

$$\{[s_1 \alpha + s_2 \beta] \in E_g(G, V) \mid \alpha \in A_0, |s_2| < \delta, \|\beta - b\| < \delta\}$$

with δ small enough.

For any $b \in S(G, V)_g$ with $\|b\| \leq 1$, let V_{b, δ_b} denote the open subset of $E_g(G, V)$

$$\{[s_1 \alpha + s_2 \beta] \in E_g(G, V) \mid s_2 > 1 - \delta_b, \|\beta - b\| < \delta_b\}$$

with δ_b small enough.

Let's consider the open neighborhood U of x that is the union of the spaces defined above

$$U := \left(\bigcup_w U_{w, \delta_w} \right) \cup W_\delta \cup \left(\bigcup_b V_{b, \delta_b} \right)$$

where w goes over all the points of the form $[t_1 c_0 + t_2 b]$ in $E_g(G, V)$ with $0 < t_2 < 1$, and b goes over all the points in $S(G, V)_g$ with $\|b\| \leq 1$.

The preimage of each U_{w, δ_w} and W_δ is open, the proof of which is analogous to Case I and II. The preimage of V_{b, δ_b} is the smash product of S^{V^g} and the open set of $S^{(V^g)^\perp}$

$$\{w_2 \in S^{(V^g)^\perp} \mid \|w_2\| > 1 - \delta_b, \|w_2 - b\| < \delta_b\},$$

thus, is open.

The preimage of U is the union of open subsets in S^V , thus, open.

Therefore, The map $\eta_g^E(G, V)$ defined in (6.18) is continuous. \square

REMARK 6.20. For any $g \in G^{tors}$, it's straightforward to check the diagram below commutes.

$$\begin{array}{ccc} S^{V^g} & \xrightarrow{\eta_g(G, V)} & F_g(G, V) \\ \downarrow & & \downarrow \\ S^V & \xrightarrow{\eta_g^E(G, V)} & E_g(G, V) \end{array}$$

where both vertical maps are inclusions.

By Lemma 6.19, the map

$$\eta^E(G, V) : S^V \longrightarrow \prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V))$$

defined by

$$(6.19) \quad v \mapsto \prod_{g \in G_{conj}^{tors}} (\alpha \mapsto \eta_g^E(G, V)(\alpha \cdot v)),$$

is well-defined and continuous. Moreover, $\eta^E : S \longrightarrow E$ with $E(G, V)$ defined in (6.13) is a well-defined functor.

Next, let's construct the multiplication map μ^E . First we define a map $\mu_{(g,h)}^E((G, V), (H, W)) :$
 $E_g(G, V) \wedge E_h(H, W) \longrightarrow E_{(g,h)}(G \times H, V \oplus W)$ by sending a point $[t_1 a_1 + t_2 b_1] \wedge$
 $[u_1 a_2 + u_2 b_2]$ to
 (6.20)

$$\begin{cases} [(1 - \sqrt{t_2^2 + u_2^2})\mu_{(g,h)}^F((G, V), (H, W))(a_1 \wedge a_2) + \sqrt{t_2^2 + u_2^2}(b_1 + b_2)], & \text{if } t_2^2 + u_2^2 \leq 1 \text{ and } t_2 u_2 \neq 0; \\ [(1 - t_2)\mu_{(g,h)}^F((G, V), (H, W))(a_1 \wedge a_2) + t_2 b_1], & \text{if } u_2 = 0 \text{ and } 0 < t_2 < 1; \\ [(1 - u_2)\mu_{(g,h)}^F((G, V), (H, W))(a_1 \wedge a_2) + u_2 b_2], & \text{if } t_2 = 0 \text{ and } 0 < u_2 < 1; \\ [1\mu_{(g,h)}^F((G, V), (H, W))(a_1 \wedge a_2) + 0], & \text{if } u_2 = 0 \text{ and } t_2 = 0; \\ x_{g,h}, & \text{Otherwise.} \end{cases}$$

where $\mu_{(g,h)}^E((G, V), (H, W))$ is the one defined in (6.11) and $x_{g,h}$ is the basepoint of $E_{(g,h)}(G \times H, V \oplus W)$.

LEMMA 6.21. *The map $\mu_{(g,h)}^E((G, V), (H, W))$ defined in (6.20) is well-defined and continuous.*

PROOF. Note that when either a_1 is the basepoint of $F_g(G, V)$, or a_2 is the basepoint of $F_h(H, W)$, or $t_2 = 1$, or $u_2 = 1$, the point $[t_1 a_1 + t_2 b_1] \wedge [u_1 a_2 + u_2 b_2]$ is mapped to the basepoint $x_{g,h}$.

The spaces $S(G, V)_g$ have the following properties:

- (i) There is no zero vector in any $S(G, V)_g$ by its construction;
- (ii) For any $b_1 \in S(G, V)_g$, $b_2 \in S(H, W)_h$, b_1 , b_2 and $b_1 + b_2$ are all in $S(G \times H, V \oplus W)_{(g,h)}$. b_1 and b_2 are orthogonal to each other, so $\|b_1 + b_2\|^2 = \|b_1\|^2 + \|b_2\|^2$. Thus, if $t_2 u_2 \neq 0$, $\|b_1 + b_2\| \leq \sqrt{t_1^2 + t_2^2}$.

Therefore, $\mu_{(g,h)}^E((G, V), (H, W))$ is well-defined.

Let

$$x = [s_1 \alpha + s_2 \beta]$$

be a point in the image of $\mu_{(g,h)}^E((G, V), (H, W))$. If s_2 is nonzero, there is unique $\beta_1 \in S(G, V)_g \cup \{0\}$ and unique $\beta_2 \in S(H, W)_h \cup \{0\}$ such that $\beta = \beta_1 + \beta_2$.

For each point in the image, I pick an open neighborhood of it so that its preimage in $E_g(G, V) \wedge E_h(H, W)$ is open.

Case I: x is not the basepoint, $0 < s_1, s_2 < 1$ and β_1 and β_2 are both nonzero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value. We consider the open neighborhood $U_{x,\delta}$ of x

$$U_{x,\delta} := \{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_1 - \beta_1\| < \delta, \|d_2 - \beta_2\| < \delta, a \in A(\alpha), |r_2^2 - s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G, V)_g \cup \{0\}$ and $d_2 \in S(H, W)_h \cup \{0\}$.

The preimage of $U_{x,\delta}$ is

$$\begin{aligned} \{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \in E_g(G, V) \wedge E_h(H, W) \mid a_1 \wedge a_2 \in \mu_{(g,h)}^F((G, V), (H, W))^{-1}(A(\alpha)), \\ \|d_1 - \beta_1\| < \delta, \|d_2 - \beta_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\}, \end{aligned}$$

where $\mu_{(g,h)}^F((G, V), (H, W))$ is the multiplication defined in (6.11).

Note that $E_g(G, V) \wedge E_h(H, W)$ is the quotient space of a subspace of the product of spaces

$$F_g(G, V) \times S(G, V)_g \times [0, 1] \times F_h(H, W) \times S(H, W)_h \times [0, 1]$$

and $U_{x,\delta}$ is the quotient of an open subset of this product. So it is open in $E_g(G, V) \wedge E_h(H, W)$.

Case II: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(H, W)_h$.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value.

Consider the open neighborhood $W_{x,\delta}$ of x

$$W_{x,\delta} := \{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_1 - \beta_1\| < \delta, \|d_2\| < \delta, a \in A(\alpha), |r_2^2 - s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G, V)_g \cup \{0\}$ and $d_2 \in S(H, W)_h \cup \{0\}$.

The preimage of $W_{x,\delta}$ is

$$\begin{aligned} \{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \in E_g(G, V) \wedge E_h(H, W) \mid a_1 \wedge a_2 \in \mu_{(g,h)}^F((G, V), (H, W))^{-1}(A(\alpha)), \\ \|d_1 - \beta_1\| < \delta, \|d_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\}. \end{aligned}$$

It is the quotient of an open subspace of the product

$$F_g(G, V) \times S(G, V)_g \times [0, 1] \times F_h(H, W) \times S(H, W)_h \times [0, 1].$$

So the preimage of $W_{x,\delta}$ is open in $E_g(G, V) \wedge E_h(H, W)$.

Case III: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(G, V)_g$.

We can show the map is continuous at such points in a way analogous to Case II.

Case IV x is not the basepoint and s_2 is zero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value.

Consider the open neighborhood of x

$$B_{x,\delta} := \{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid a \in A(\alpha), \|d_1\| < \delta, \|d_2\| < \delta, 0 \leq r_2^2 < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G, V)_g \cup \{0\}$ and $d_2 \in S(H, W)_h \cup \{0\}$.

The preimage of $B_{x,\delta}$ is

$$\begin{aligned} \{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \in E_g(G, V) \wedge E_h(H, W) \mid a_1 \wedge a_2 \in \mu_{(g,h)}^F((G, V), (H, W))^{-1}(A(\alpha)), \\ \|d_1\| < \delta, \|d_2\| < \delta, 0 \leq t_2^2 + u_2^2 < \delta\}. \end{aligned}$$

It is the quotient of an open subspace of the product

$$F_g(G, V) \times S(G, V)_g \times [0, 1] \times F_h(H, W) \times S(H, W)_h \times [0, 1].$$

So the preimage of $B_{x,\delta}$ is open in $E_g(G, V) \wedge E_h(H, W)$.

Case V: $x = [s_1 \alpha + s_2 \beta]$ is the base point.

Let $A_0(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$.

For any point w in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and b_1, b_2 both nonzero, let U_{w,δ_w} be the open subset of $E_{(g,h)}(G \times H, V \oplus W)$

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_1 - b_1\| < \delta_w, \|d_2 - b_2\| < \delta_w, a \in A_0(\alpha), |r_2^2 - t_2^2| < \delta_w\}$$

with δ_w small enough.

For each point y in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and $b \in S(H, W)_h$, let W_{y,δ_y} be the open subset

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_1 - b_1\| < \delta_y, \|d_2\| < \delta_y, a \in A_0(\alpha), |r_2^2 - t_2^2| < \delta_y\}$$

with δ_y small enough.

For each point z in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and $b \in S(G, V)_g$, let V_{z, δ_z} be the open subset

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_2 - b_2\| < \delta_z, \|d_1\| < \delta_z, a \in A_0(\alpha), |r_2^2 - t_2^2| < \delta_z\}$$

with δ_z small enough.

Let $B_{x_0, \delta}$ denote the open set

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d_2\| < \delta, \|d_1\| < \delta, a \in A_0(c_0), 0 \leq r_2 < \delta\}$$

with δ small enough,

For each θ in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $0 + 1b$, let $D_{\theta, \delta_\theta}$ be the open subset

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) \mid \|d - b\| < \delta_\theta, 1 \geq r_2 \geq 1 - \delta_\theta\}$$

with δ_θ small enough.

Let's consider the open neighborhood of x in $E_{(g,h)}(G \times H, V \oplus W)$ that is the union of the spaces above

$$U := \left(\bigcup_w U_{w, \delta_w} \right) \cup \left(\bigcup_y W_{y, \delta_y} \right) \cup \left(\bigcup_z V_{z, \delta_z} \right) \cup B_{x_0, \delta} \cup \left(\bigcup_\theta D_{\theta, \delta_\theta} \right)$$

where w goes over all the points in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and b_1, b_2 both nonzero, y goes over all the points in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and $b \in S(H, W)_h$, z goes over all the points in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and $b \in S(G, V)_g$, and θ goes over all the points of the form $0 + 1b$ in $E'_{(g,h)}(G \times H, V \oplus W)$.

The preimage of each $U_{x, \delta_x}, W_{y, \delta_y}, V_{z, \delta_z}, B_{x_0, \delta}$ is open, the proof of which are analogous to that of Case I, II, III and IV. The preimage of $D_{\theta, \delta_\theta}$ is

$$\{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \in E_g(G, V) \wedge E_h(H, W) \mid \|d_1 + d_2 - b\| < \delta_\theta, 1 - \sqrt{t_2^2 + u_2^2} < \delta_\theta\},$$

which is open. Therefore, the preimage of U is open.

Combining all the cases above, the multiplication $\mu_{(g,h)}^E((G, V), (H, W))$ defined in (6.20) is continuous. \square

REMARK 6.22. For any $g \in G^{tors}, h \in H^{tors}$, we have the diagram below commutes.

$$\begin{array}{ccc} F_g(G, V) \wedge F_h(H, W) & \xrightarrow{\mu_{(g,h)}^F((G, V), (H, W))} & F_{(g,h)}(G \times H, V \oplus W) \\ \downarrow & & \downarrow \\ E_g(G, V) \wedge E_h(H, W) & \xrightarrow{\mu_{(g,h)}^E((G, V), (H, W))} & E_{(g,h)}(G \times H, V \oplus W) \end{array}$$

where the vertical maps are both inclusion into the end of the join.

The basepoint of $E(G, V)$ is the product of the basepoint of each factor $\text{Map}_{CG(g)}(G, E_g(G, V))$, i.e. the product of the constant map to the base point of each $E_g(G, V)$.

We can define the multiplication

$$(6.21) \quad \mu^E((G, V), (H, W)) : E(G, V) \wedge E(H, W) \longrightarrow E(G \times H, V \oplus W)$$

by the composition

$$\begin{aligned}
& \prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V)) \wedge \prod_{h \in H_{conj}^{tors}} \text{Map}_{C_H(h)}(H, E_h(H, W)) \longrightarrow \\
& \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V)) \wedge \text{Map}_{C_H(h)}(H, E_h(H, W)) \longrightarrow \\
& \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \text{Map}_{C_{G \times H}(g, h)}(G \times H, E_g(G, V) \wedge E_h(H, W)) \longrightarrow \\
& \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \text{Map}_{C_{G \times H}(g, h)}(G \times H, E_{(g, h)}(G \times H, V \oplus W))
\end{aligned}$$

where the first map sends

$$\prod_{g \in G_{conj}^{tors}} \alpha_g \wedge \prod_{h \in H_{conj}^{tors}} \beta_h$$

to

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \alpha_g \wedge \beta_h,$$

the second map sends a point

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \alpha_g \wedge \beta_h$$

to

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((a, b) \mapsto \alpha_g(a) \wedge \beta_h(b) \right),$$

the third map sends

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} f_{(g, h)}$$

to

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \mu_{(g, h)}^E((G, V), (H, W)) \circ f_{(g, h)}.$$

More explicitly, $\mu^E((G, V), (H, W))$ sends

$$\left(\prod_{g \in G_{conj}^{tors}} \alpha_g \right) \wedge \left(\prod_{h \in H_{conj}^{tors}} \beta_h \right)$$

to

$$(6.22) \quad \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((g', h') \mapsto \mu_{(g, h)}^E((G, V), (H, W))(\alpha_g(g') \wedge \beta_h(h')) \right).$$

LEMMA 6.23. *Let G, H, K be compact Lie groups. Let V be an orthogonal G -representation, W an orthogonal H -representation, and U an orthogonal K -representation. Let $g \in G^{tors}$, $h \in H^{tors}$, and $k \in K^{tors}$. Then we have the commutative diagrams below.*

$$(6.23) \quad \begin{array}{ccc} S^V \wedge S^W & \xrightarrow{\eta_g^E(G,V) \wedge \eta_h^E(H,W)} & E_g(G,V) \wedge E_h(H,W) \\ \downarrow \cong & & \downarrow \mu_{(g,h)}^E((G,V),(H,W)) \\ S^{V \oplus W} & \xrightarrow{\eta_{(g,h)}^E(G \times H, V \oplus W)} & E_{(g,h)}(G \times H, V \oplus W) \end{array}$$

$$(6.24) \quad \begin{array}{ccc} E_g(G,V) \wedge E_h(H,W) \wedge E_k(K,U) & \xrightarrow{\mu_g^E((G,V),(H,W)) \wedge Id} & E_{(g,h)}(G \times H, V \oplus W) \wedge E_k(K,U) \\ \downarrow Id \wedge \mu_{(h,k)}^E(H \times K, W \oplus U) & & \downarrow \mu_{(g,h,k)}^E((G \times H, V \oplus W), (K,U)) \\ E_g(G,V) \wedge E_{(h,k)}(H \times K, W \oplus U) & \xrightarrow{\mu_{(g,(h,k))}^E((G,V),(H \times K, W \oplus U))} & E_{(g,h,k)}(G \times H \times K, V \oplus W \oplus U) \end{array}$$

$$(6.25) \quad \begin{array}{ccc} S^V \wedge E_h(H,W) & \xrightarrow{\eta_g^E(G,V) \wedge Id} E_g(G,V) \wedge E_h(H,W) & \xrightarrow{\mu_{(g,h)}^E((G,V),(H,W))} E_{(g,h)}(G \times H, V \oplus W) \\ \downarrow \tau & & \downarrow E_{(g,h)}(\tau) \\ E_h(H,W) \wedge S^V & \xrightarrow{Id \wedge \eta_g^E(G,V)} E_h(H,W) \wedge E_g(G,V) & \xrightarrow{\mu_{(h,g)}^E((H,W),(G,V))} E_{(h,g)}(H \times G, W \oplus V) \end{array}$$

Moreover, we have

$$(6.26) \quad \mu_{(g,h)}^E((G,V),(H,W))(x \wedge y) = \mu_{(h,g)}^E((H,W),(G,V))(y \wedge x)$$

for any $x \in E_g(G,V)$ and $y \in E_h(H,W)$.

PROOF. In this proof, I identify the end $F_g(G,V)$ in the space $E_g(G,V)$ with the points of the form $(a, 0, 0)$, i.e. $1a + 00$, in the space (6.17) as indicated in Remark 6.17. And if the coordinate t_2 in a point $t_1a + t_2b$ is zero, then b is the zero vector.

(i) Unity.

Let $v \in S^V$ and $w \in S^W$. Let

$$v = v_1 \wedge v_2, \text{ with } v_1 \in S^{V^g}, \text{ and } v_2 \in S^{(V^g)^\perp},$$

$$w = w_1 \wedge w_2, \text{ with } w_1 \in S^{W^h}, \text{ and } w_2 \in S^{(W^h)^\perp},$$

$$\mu_{(g,h)}^E((G,V),(H,W)) \circ (\eta_g^E(G,V) \wedge \eta_h^E(H,W))(v \wedge w)$$

is the basepoint if $\|v_2\|^2 + \|w_2\|^2 \geq 1$. If $\|v_2\|^2 + \|w_2\|^2 \leq 1$, it equals

$$(6.27) \quad [(1 - \sqrt{\|v_2\|^2 + \|w_2\|^2})\eta_g(G,V)(v_1) \wedge \eta_h(H,W)(w_1) + \sqrt{\|v_2\|^2 + \|w_2\|^2}(v_2 + w_2)].$$

On the other direction, $\eta_{(g,h)}^E(G \times H, V \oplus W)(v \wedge w)$ is the basepoint if $\|v_2 + w_2\| \geq 1$. Note that since v_2 and w_2 are orthogonal to each other, $\|v_2 + w_2\|^2 = \|v_2\|^2 + \|w_2\|^2$.

If $\|v_2 + w_2\| \leq 1$, it is

$$(6.28) \quad [(1 - \sqrt{\|v_2\|^2 + \|w_2\|^2})\eta_g(G,V)(v_1) \wedge \eta_h(H,W)(w_1) + \sqrt{\|v_2\|^2 + \|w_2\|^2}(v_2 + w_2)],$$

which is equal to the term in (6.27) by Proposition 6.13 (ii).

(ii) Associativity.

Let $x = [t_1 a_1 + t_2 b_1]$ be a point in $E_g(G, V)$, $y = [s_1 a_2 + s_2 b_2]$ a point in $E_h(H, W)$, and $z = [r_1 a_3 + r_2 b_3]$ a point in $E_k(K, U)$.

$$\mu_{((g,h),k)}^E((G \times H, V \oplus W), (K, U)) \circ (\mu_{(g,h)}^E((G, V), (H, W)) \wedge Id)(x \wedge y \wedge z)$$

is the basepoint if $t_2^2 + s_2^2 + r_2^2 \geq 1$.

If $t_2^2 + s_2^2 + r_2^2 \leq 1$,

$$\begin{aligned} & \mu_{((g,h),k)}^E((G \times H, V \oplus W), (K, U)) \circ (\mu_{(g,h)}^E((G, V), (H, W)) \wedge Id)(x \wedge y \wedge z) \\ &= \mu_{((g,h),k)}^E((G \times H, V \oplus W), (K, U)) \\ & \quad ([(1 - \sqrt{t_2^2 + s_2^2}) \mu_{g,h}^F((G, V), (H, W))(a_1 \wedge a_2) + \sqrt{t_2^2 + s_2^2} (b_1 + b_2)] \wedge z) \\ &= [(1 - \sqrt{t_2^2 + s_2^2 + r_2^2}) \mu_{((g,h),k)}^F((G \times H, V \oplus W), (K, U)) (\mu_{g,h}^F((G, V), (H, W))(a_1 \wedge a_2) \wedge a_3) \\ & \quad + \sqrt{t_2^2 + s_2^2 + r_2^2} (b_1 + b_2 + b_3)] \end{aligned}$$

Note that

$$(\sqrt{t_2^2 + s_2^2})^2 + u_2^2 = (\sqrt{t_2^2 + s_2^2 + u_2^2})^2$$

Then let's consider the other direction.

$$\mu_{(g,(h,k))}^E((G, V), (H \times K, W \oplus U)) \circ (Id \wedge \mu_{(h,k)}^E(H \times K, W \oplus U))(x \wedge y \wedge z)$$

is the basepoint if $t_2^2 + s_2^2 + r_2^2 \geq 1$.

If $t_2^2 + s_2^2 + r_2^2 \leq 1$,

$$\begin{aligned} & \mu_{(g,(h,k))}^E((G, V), (H \times K, W \oplus U)) \circ (Id \wedge \mu_{(h,k)}^E(H \times K, W \oplus U))(x \wedge y \wedge z) \\ &= \mu_{(g,(h,k))}^E((G, V), (H \times K, W \oplus U)) \\ & \quad (x \wedge [(1 - \sqrt{r_2^2 + s_2^2}) \mu_{(h,k)}^F((H, W), (K, U))(a_2 \wedge a_3) + \sqrt{r_2^2 + s_2^2} (b_2 + b_3)]) \\ &= [(1 - \sqrt{t_2^2 + s_2^2 + r_2^2}) \mu_{(g,(h,k))}^F((G, V), (H \times K, W \oplus U)) (a_1 \wedge \mu_{(h,k)}^F((H, W), (K, U))(a_2 \wedge a_3)) \\ & \quad + \sqrt{t_2^2 + s_2^2 + r_2^2} (b_1 + b_2 + b_3)], \end{aligned}$$

which is equal to

$$\begin{aligned} & [(1 - \sqrt{t_2^2 + s_2^2 + r_2^2}) \mu_{((g,h),k)}^F((G \times H, V \oplus W), (K, U)) (\mu_{(g,h)}^F((G, V), (H, W))(a_1 \wedge a_2) \wedge a_3) \\ & \quad + \sqrt{t_2^2 + s_2^2 + r_2^2} (b_1 + b_2 + b_3)] \end{aligned}$$

by Proposition 6.13 (ii).

(iii) Centrality of unit.

Let $v \in S^V$ and $x = [t_1 a + t_2 b]$ a point in $E_h(H, W)$.

$$E_{(g,h)}(\tau) \circ \mu_{(g,h)}^E((G, V), (H, W)) \circ (\eta_g^E(G, V) \wedge Id)(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geq 1$. If $\|v_2\|^2 + t_2^2 \leq 1$, it's

$$\begin{aligned} & [(1 - \sqrt{\|v_2\|^2 + t_2^2}) \mu_{(g,h)}^F((G, V), (H, W)) (\eta_g(G, V)(v_1) \wedge a) + \sqrt{\|v_2\|^2 + t_2^2} (v_2 + b)] \\ &= [(1 - \sqrt{\|v_2\|^2 + t_2^2}) \mu_{(h,g)}^F((H, W), (G, V)) (a \wedge \eta_g(G, V)(v_1)) + \sqrt{\|v_2\|^2 + t_2^2} (v_2 + b)], \end{aligned}$$

by Proposition 6.13 (ii).

$$\mu_{(h,g)}^E((H, W), (G, V)) \circ (Id \wedge \eta_h^E(H, W)) \circ \tau(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geq 1$. If $\|v_2\|^2 + t_2^2 \leq 1$, it's

$$[(1 - \sqrt{\|v_2\|^2 + t_2^2})\mu_{(h,g)}^F((H, W), (G, V))(a \wedge \eta_g(G, V)(v_1)) + \sqrt{\|v_2\|^2 + t_2^2}(v_2 + b)].$$

So the centrality of unit diagram commutes.

(iv) According to the formula of $\mu_{(g,h)}^E((G, V), (H, W))$ and Proposition 6.13 (iv),

$$\begin{aligned} \mu_{(g,h)}^E((G, V), (H, W))(x \wedge y) &= [(1 - \sqrt{t_2^2 + s_2^2})\mu_{g,h}^F((G, V), (H, W))(a_1 \wedge a_2) + \\ &\sqrt{t_2^2 + s_2^2}(b_1 + b_2)] = [(1 - \sqrt{t_2^2 + s_2^2})\mu_{h,g}^F((H, W), (G, V))(a_2 \wedge a_1) + \sqrt{t_2^2 + s_2^2}(b_2 + \\ &b_1)] = \mu_{(h,g)}^E((H, W), (G, V))(y \wedge x). \end{aligned}$$

□

THEOREM 6.24. *Let $\Delta_G : G \rightarrow G \times G$ be the diagonal map $g \mapsto (g, g)$. For G -representations V and W , let*

$$(\Delta_G)_{V \oplus W}^* : E(G \times G, V \oplus W) \rightarrow E(G, V \oplus W)$$

denote the restriction map defined by the formula (6.46). Then

$$E : \mathcal{I}_G \rightarrow G\mathcal{T}$$

together with the unit map η^E defined in (6.19) and the multiplication $\Delta_G^ \circ \mu^E((G, -), (G, -))$ gives a commutative \mathcal{I}_G -FSP that weakly represents $QEll_G^*(-)$.*

PROOF. Let G, H, K be compact Lie groups, V an orthogonal G -representation, W an orthogonal H -representation and U an orthogonal K -representation.

Let

$$X = \prod_{g \in G_{conj}^{tors}} \alpha_g \in E(G, V); Y = \prod_{h \in H_{conj}^{tors}} \beta_h \in E(H, W); Z = \prod_{k \in K_{conj}^{tors}} \gamma_k \in E(K, U).$$

First let's check the diagram of unity commutes.

Let $v \in S^V$ and $w \in S^W$. $\mu^E((G, V), (H, W)) \circ (\eta^E(G, V) \wedge \eta^E(H, W))(v \wedge w)$ is

$$(6.29) \quad \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((g', h') \mapsto \mu_{(g,h)}^E((G, V), (H, W)) \circ (\eta_g^E(G, V) \wedge \eta_h^E(H, W))(g' \cdot v \wedge h' \cdot w) \right).$$

$\eta^E(G \times H, V \oplus W)(v \wedge w)$ is

$$(6.30) \quad \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((g', h') \mapsto \eta_{(g,h)}^E(G \times H, V \oplus W)(g' \cdot v \wedge h' \cdot w) \right),$$

which is equal to the term (6.29) by Lemma 6.23.

Next let's check the diagram of associativity commutes.

$$\mu^E((G \times H, V \oplus W), (K, U)) \circ (\mu^E((G, V), (H, W)) \wedge Id)(X \wedge Y \wedge Z)$$

is

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}, k \in G_{conj}^{tors}} \left((g', h', k') \mapsto \mu_{((g,h),k)}^E((G \times H, V \oplus W), (K, U)) \circ (\mu_{(g,h)}^E((G, V), (H, W)) \wedge Id)(\alpha_g(g') \wedge \beta_h(h') \wedge \gamma_k(k')) \right)$$

And

$$\mu^E((G, V), (H \times K, W \oplus U)) \circ (Id \wedge \mu^E(H \times K, W \oplus U))(X \wedge Y \wedge Z)$$

is

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}, k \in G_{conj}^{tors}} \left((g', h', k') \mapsto \mu_{(g,(h,k))}^E((G, V), (H \times K, W \oplus U)) \circ (Id \wedge \mu_{(h,k)}^E(H \times K, W \oplus U))(\alpha_g(g') \wedge \beta_h(h') \wedge \gamma_k(k')) \right)$$

By Lemma 6.23, the two terms are equal.

Then let's check the diagram of centrality of unit commutes.

$$E(\tau) \circ \mu^E((G, V), (H, W)) \circ (\eta^E(G, V) \wedge Id)(v \wedge X)$$

is

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((h', g') \mapsto E_{(g,h)}(\tau) \circ \mu_{(g,h)}^E((G, V), (H, W)) \circ (\eta_g^E(G, V) \wedge Id)((g' \cdot v) \wedge \beta_h(h')) \right)$$

$$\mu^E((H, W), (G, V)) \circ (Id \wedge \eta^E(H, W)) \circ \tau(v \wedge X)$$

is

$$\prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((h', g') \mapsto \mu_{(h,g)}^E((H, W), (G, V)) \circ (Id \wedge \eta_h^E(H, W)) \circ \tau((g' \cdot v) \wedge \beta_h(h')) \right)$$

The two terms are equal. So the centrality of unit diagram commutes.

Moreover, let's check

$$(6.31) \quad \mu^E((G, V), (H, W))(X \wedge Y) = \mu^E((H, W), (G, V))(Y \wedge X).$$

By Lemma 6.23,

$$\begin{aligned} \mu^E((G, V), (H, W))(X \wedge Y) &= \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((g', h') \mapsto \mu_{(g,h)}^E((G, V), (H, W))(\alpha_g(g') \wedge \beta_h(h')) \right) \\ &= \prod_{g \in G_{conj}^{tors}, h \in H_{conj}^{tors}} \left((h', g') \mapsto \mu_{(h,g)}^E((H, W), (G, V))(\beta_h(h') \wedge \alpha_g(g')) \right) \\ &= \mu^E((H, W), (G, V))(Y \wedge X). \end{aligned}$$

Therefore we have the commutativity of E .

□

REMARK 6.25. For each $g \in G_{conj}^{tors}$, consider

$$E(G, V)_g := \text{Map}_{CG(g)}(G, E_g(G, V)).$$

It has the relation with $E(G, V)$

$$E(G, V) = \prod_{g \in G_{conj}^{tors}} E(G, V)_g.$$

Moreover, we can define a unit map $\eta_g^{E'}(G, V)$ and a multiplication map $\mu_{(g,h)}^{E'}((G, V), (H, W))$.

Let

$$\eta_g^{E'}(G, V) : S^V \longrightarrow E(G, V)_g$$

be the map defined by

$$(6.32) \quad v \mapsto (\alpha \mapsto \eta_g^E(G, V)(\alpha \cdot v)).$$

Let $\mu_{(g,h)}^{E'}((G, V), (H, W)) : E(G, V)_g \wedge E(H, W)_h \longrightarrow E(G \times H, V \oplus W)_{(g,h)}$ denote the map sending

$$\alpha_g \wedge \beta_h$$

to

$$(6.33) \quad (g', h') \mapsto \mu_{(g,h)}^E((G, V), (H, W))(\alpha_{g'}(g') \wedge \beta_{h'}(h')).$$

We have the relations

$$\eta^E(G, V) = \prod_{g \in G_{conj}^{tors}} \eta_g^{E'}(G, V), \quad \mu^E((G, V), (H, W)) = \prod_{g \in G_{conj}^{tors}} \mu_g^{E'}((G, V), (H, W)).$$

The proof of Theorem 6.24 also shows that $E(G, -)_g$ is a \mathcal{I}_G -FSP with the unit map $\eta_g^{E'}(G, -)$ and the multiplication $\Delta_G^* \circ \mu_{(g,g)}^{E'}((G, -), (G, -))$.

REMARK 6.26. An orthogonal spectrum can give orthogonal G -spectra for each G . Then we may ask under what condition a given orthogonal G -spectrum is part of a global family, i.e. arise from an orthogonal spectrum. There is a criterion. The two conditions below are equivalent:

- (a) The G -spectrum Y is isomorphic to an orthogonal G -spectrum of the form $X \langle G \rangle$ for some orthogonal spectrum X ;
- (b) for every trivial G -representation V the G -action on $Y(V)$ is trivial.

It's straightforward to check E doesn't satisfy the condition (b), so it cannot arise from an orthogonal spectrum.

PROPOSITION 6.27. Let G be any compact Lie group. Let V be an ample orthogonal G -representation and W an orthogonal G -representation.

Let $\sigma_{G,V,W}^E : S^W \wedge E(G, V) \longrightarrow E(G, V \oplus W)$ denote the structure map of E defined by the unit map $\eta^E(G, V)$. Let $\tilde{\sigma}_{G,V,W}^E$ denote the right adjoint of $\sigma_{G,V,W}^E$. Then

$$\tilde{\sigma}_{G,V,W}^E : E(G, V) \longrightarrow \text{Map}(S^W, E(G, V \oplus W))$$

is a G -weak equivalence.

PROOF. From the formula of $\eta^E(G, V)$, we can get an explicit formula for

$$\tilde{\sigma}_{G,V,W}^E : E(G, V) \longrightarrow \text{Map}(S^W, E(G, V \oplus W)).$$

For any element

$$\alpha := \prod_{g \in G_{conj}^{tors}} \alpha_g$$

in

$$E(G, V) = \prod_{g \in G_{conj}^{tors}} \text{Map}_{C_G(g)}(G, E_g(G, V)).$$

Let w be an element in S^W . For each $g \in G_{conj}^{tors}$, w has a unique decomposition

$$w = w_g^1 \wedge w_g^2$$

with $w_g^1 \in S^{W^g}$ and $w_g^2 \in S^{(W^g)^\perp}$. $\tilde{\sigma}_{G,V,W}^E$ sends α to

$$w \mapsto \left(\prod_{g \in G_{conj}^{tors}} g' \mapsto \Delta_G^* \circ \mu_{(g,g)}^E((G, V), (G, W))(\alpha_g(g'), \eta_g^E(G, W)(g' \cdot w)) \right).$$

It suffices to show that for each $g \in G_{conj}^{tors}$, the map

(6.34)

$$\begin{aligned} \tilde{\sigma}_{G,g,V,W}^E : E_g(G, V) &\longrightarrow \text{Map}_{C_G(g)}(S^W, E_g(G, V \oplus W)) \\ (6.35) \quad x &\mapsto \left(w \mapsto \Delta_G^* \circ \mu_{(g,g)}^E((G, V), (G, W))(x, \eta_g^E(G, W)(w)) \right) \end{aligned}$$

is a $C_G(g)$ -weak equivalence.

I check for each closed subgroup H of $C_G(g)$, the map $(\tilde{\sigma}_{G,g,V,W}^E)^H$ on the fixed point space is a homotopy equivalence.

Case I: $g \in H$.

$E_g(G, V)^H$ is the space $F_g(G, V)^H$. By Proposition 6.13,

$$\tilde{\sigma}_g(G, V, W)^H : F_g(G, V)^H \longrightarrow \text{Map}_H(S^{W^g}, F_g(G, V \oplus W))$$

is a weak equivalence.

By Theorem 4.6,

$$\text{Map}_H(S^W, E_g(G, V \oplus W)) \longrightarrow \text{Map}_H(S^{W^g}, F_g(G, V \oplus W)), \quad f \mapsto f|_{S^{W^g}}$$

is a homotopy equivalence.

And we have the diagram below whose commutativity can be checked directly by applying the formula (6.35).

$$(6.36) \quad \begin{array}{ccc} F_g(G, V)^H & \xrightarrow{\simeq} & \text{Map}_H(S^{W^g}, F_g(G, V \oplus W)) \\ & \searrow & \uparrow \simeq \\ & & \text{Map}_H(S^W, E_g(G, V \oplus W)) \end{array}$$

So

$$\tilde{\sigma}_{G,g,V,W}^E : F_g(G, V)^H \longrightarrow \text{Map}_H(S^{W^g}, F_g(G, V \oplus W))$$

is a homotopy equivalence.

Case II: g is not in H .

In this case, $E_g(G, V)^H$ is contractible. It suffices to show that $\text{Map}_H(S^W, E_g(G, V \oplus W))$ is also contractible.

Note that for any closed subgroup H' of H , $E_g(G, V \oplus W)^{H'}$ is contractible. So for each n -cell $H/H' \times D^n$ of S^W , it's mapped to $E_g(G, V \oplus W)^{H'}$ unique up to homotopy.

So $\text{Map}_H(S^W, E_g(G, V \oplus W))$ is contractible.

Therefore $\tilde{\sigma}_{G,g,V,W}^E$ is a $C_G(g)$ -weak equivalence. So $\tilde{\sigma}_{G,V,W}^E$ is a G -weak equivalence. \square

By Proposition 6.15 and Proposition 6.27 we can get the conclusion below.

COROLLARY 6.28. For any compact Lie group G , any faithful G -representation V , E represents $QEll_G^V(-)$ weakly in the sense

$$(6.37) \quad \pi_k(E(G, V)) = QEll_G^V(S^k).$$

REMARK 6.29. The readers may have realized that the construction of the G -FSP E is not related to the concrete construction of the global K-theory. In other words, for any ultra commutative global ring spectrum X for a global cohomology theory H , we can form a \mathcal{I}_G -FSP that weakly represents the cohomology theory

$$X \mapsto \prod_{\sigma \in G_{conj}^{tors}} H_{\Lambda(\sigma)}^*(X^\sigma).$$

6.5. The Restriction Map. Let G be a compact Lie group. For any complex inner product space V , $KU(V)$ has an $O(V)$ -action inheriting from that on V . Any real representation $\rho : G \rightarrow O(V)$ defines a G -action on $E(G, V)$. In this section I construct the restriction maps $E(G, V) \rightarrow E(H, V)$ for group homomorphisms $H \rightarrow G$.

Let $\phi : H \rightarrow G$ be a group homomorphism and V a G -representation. From the change of group isomorphism, for any homomorphism of compact Lie groups $\phi : H \rightarrow G$ and H -space X , we have

$$QEll_G^*(G \times_H X)$$

isomorphic to

$$QEll_H^*(X).$$

Thus, for any subgroup K of H , we have the isomorphism

$$QEll_G^n(G/K) = QEll_G^n(G \times_H H/K) \cong QEll_H^n(H/K).$$

So by Proposition 6.15 the space $E(G, V)^K$ is homotopy equivalent to $E(H, V)^K$ when V is a faithful G -representation. It implies when we consider $E(G, V)$ as an H -space, it is H -weak equivalent to $E(H, V)$.

As indicated in Remark 6.26, the orthogonal G -spectrum $E(G, -)$ cannot arise from an orthogonal spectrum. As a result, the restriction map $E(G, V) \rightarrow E(H, V)$ cannot be a homeomorphism. We construct in this section a restriction map ϕ_V^* that is H -weak equivalence such that the diagram below commutes.

$$(6.38) \quad \begin{array}{ccc} \pi_k(E(G, V)) & \xrightarrow{\cong} & QEll_G^V(S^k) \\ \downarrow \pi^k(\phi_V^*) & & \downarrow \phi^* \\ \pi_k(E(H, V)) & \xrightarrow{\cong} & QEll_H^V(S^k) \end{array}$$

where ϕ^* is the restriction map of quasi-elliptic cohomology.

Now let's start the construction of the restriction map ϕ_V^* . Let X be a G -space. Let $g \in G^{tors}$ and $h \in H^{tors}$.

The group homomorphism $\phi : H \rightarrow G$ sends $C_H(h)$ to $C_G(g)$ and also gives

$$\phi_* : \Lambda_H(h) \rightarrow \Lambda_G(\phi(h)), [h', t] \mapsto [\phi(h'), t].$$

ϕ induces an H -action on X . Especially, $X^g = X^h$ and ϕ_* induces a $\Lambda_H(h)$ -action on it for each $h \in H^{tors}$.

We can define a homotopy equivalence

$$P_g(G, V) : \text{Map}_G(X, \text{Map}_{C_G(g)}(G, E_g(G, V))) \rightarrow \text{Map}_{C_G(g)}(X^g, F_g(G, V))$$

similar to that defined in (4.13), as shown below.

Let

$$\tilde{f} : X \rightarrow \text{Map}_{C_G(g)}(G, E_g(G, V))$$

be a G -map. $P_g(G, V)(\tilde{f})$ is defined as the composition

$$(6.39) \quad X^g \xrightarrow{f^g} \text{Map}_{C_G(g)}(G, E_g(G, V)) \xrightarrow{\alpha \mapsto \alpha(e)} F_g(G, V)$$

Let's first consider the equivalent definition of quasi-elliptic cohomology

$$QEll_G^*(X) = \prod_{g \in G^{tors}} K_{\Lambda_G(g)}^*(X^g)$$

with the product over all the torsion elements of G . With this definition, the restriction map can have a relatively simple form.

For each $g \in G^{tors}$, we first define the map

$$Res_{\phi, g} : \text{Map}_{C_G(g)}(G, E_g(G, V)) \rightarrow \prod_{\tau} \text{Map}_{C_H(\tau)}(H, E_{\tau}(H, V))$$

in the form

$$\prod_{\tau} \left(R_{\phi, \tau} : \text{Map}_{C_G(g)}(G, E_g(G, V)) \rightarrow \text{Map}_{C_H(\tau)}(H, E_{\tau}(H, V)) \right)$$

where τ goes over all the elements τ in H^{tors} such that $\phi(\tau) = g$. Then we will combine all the $Res_{\phi, g}$ s to define the restriction map ϕ_V^* .

The restriction map

$$\phi_V^* : E(G, V) \rightarrow E(H, V)$$

to be defined should make the diagram (6.40) commute, which implies that (6.38) commutes.

$$(6.40) \quad \begin{array}{ccccccc} X^g & \longrightarrow & X & \xrightarrow{\tilde{f}} & \text{Map}_{C_G(g)}(G, E_g(G, V)) & \xrightarrow{\alpha \mapsto \alpha(e)} & F_g(G, V) \\ \downarrow & & \downarrow & & \downarrow R_{\phi, \tau} & & \downarrow \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} \\ X^\tau & \longrightarrow & X & \xrightarrow{R_{\phi, \tau} \circ \tilde{f}} & \text{Map}_{C_H(\tau)}(H, E_\tau(H, V)) & \xrightarrow{\beta \mapsto \beta(e)} & F_\tau(H, V) \end{array}$$

where $\text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)}$ is the restriction map defined in (6.41).

Let $\tau \in H^{\text{tors}}$ and $g = \phi(h)$. Then we have the isomorphism

$$a_\tau : (V)_g^{\mathbb{R}} \oplus V^g \longrightarrow (V)_\tau^{\mathbb{R}} \oplus V^\tau$$

sending v to v . For any $[b, t] \in \Lambda_H(h)$, $a_\tau([\phi(b), t]v) = [b, t]a_\tau(v)$.

In addition, we have the restriction map $\text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} : F_g(G, V) \longrightarrow F_\tau(H, V)$ defined as below. Let $\beta : S^{(V)_g^{\mathbb{R}}} \longrightarrow KU((V)_g^{\mathbb{R}} \oplus V^g)$ be a \mathbb{R} -equivariant map. Note that $S^{(V)_\tau^{\mathbb{R}}}$ and $S^{(V)_g^{\mathbb{R}}}$ have the same underlying space, and $(V)_g^{\mathbb{R}} \oplus V^g$ and $(V)_\tau^{\mathbb{R}} \oplus V^\tau$ have the same underlying vector space.

$\text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)}(\beta)$ is defined to be the composition

$$(6.41) \quad S^{(V)_\tau^{\mathbb{R}}} \xrightarrow{x \mapsto x} S^{(V)_g^{\mathbb{R}}} \xrightarrow{\beta} KU((V)_g^{\mathbb{R}} \oplus V^g) \xrightarrow{KU(a_\tau)} KU((V)_\tau^{\mathbb{R}} \oplus V^\tau)$$

which is the identity map on the underlying spaces.

Let $\psi : K \longrightarrow H$ be another group homomorphism and $\psi(k) = h$ for some $k \in K$. Then we have

$$(6.42) \quad \text{res}|_{\Lambda_K(k)}^{\Lambda_H(h)} \circ \text{res}|_{\Lambda_H(h)}^{\Lambda_G(g)} = \text{res}|_{\Lambda_K(k)}^{\Lambda_G(g)}$$

Note $S(G, V)_g$ has the same underlying space as $S(H, V)_\tau$. Consider the join of maps

$$(6.43) \quad \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} * \text{Id} : F_g(G, V) * S(G, V)_g \longrightarrow F_\tau(H, V) * S(H, V)_\tau$$

It is the identity map on the underlying space and has the equivariant property: for any $a \in C_H(\tau)$, $x \in H$,

$$(6.44) \quad \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} * \text{Id}(\phi(a) \cdot x) = a \cdot \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} * \text{Id}(x).$$

$\text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} * b_\tau$ gives a well-defined map on the quotient space $E_g(G, V) \longrightarrow E_\tau(H, V)$. Let's use $r_{\phi, \tau}$ to denote this map. It also has the equivariant property as (6.44).

For any ρ in $\text{Map}_{C_G(g)}(G, E_g(G, V))$, let $R_{\phi, \tau}(\rho)$ be the composition

$$(6.45) \quad H \xrightarrow{\phi} G \xrightarrow{\rho} E_g(G, V) \xrightarrow{r_{\phi, \tau}} E_\tau(H, V).$$

$R_{\phi, \tau}(\rho)$ is $C_H(\tau)$ -equivariant:

$$\begin{aligned} R_{\phi, \tau}(\rho)(ah) &= r_{\phi, \tau}(\rho(\phi(ah))) = r_{\phi, \tau}(\rho(\phi(a)\phi(h))) \\ &= ar_{\phi, \tau}(\rho(\phi(h))) = a \cdot R_{\phi, \tau}(\rho)(h), \end{aligned}$$

for any $a \in C_H(\tau)$, $h \in H$.

For any $g \in \text{Im}\phi$, $\text{Res}_{\phi,g}$ is defined to be

$$\prod_{\tau} R_{\phi,\tau}$$

where τ goes over all the $\tau \in H^{\text{tors}}$ such that $\phi(\tau) = g$. The restriction map is defined to be

$$(6.46) \quad \phi_V^* := \prod_g \text{Res}_g : E(G, V) \longrightarrow E(H, V)$$

where g goes over all the elements in G^{tors} in the image of ϕ .

LEMMA 6.30. (i) R_τ defined in (6.45) is the restriction map making the diagram

$$(6.47) \quad \begin{array}{ccc} \text{Map}_{C_G(g)}(G, E_g(G, V)) & \xrightarrow{\alpha \mapsto \alpha(e)} & F_g(G, V) \\ R_{\phi,\tau} \downarrow & & \downarrow \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} \\ \text{Map}_{C_H(\tau)}(H, E_\tau(H, V)) & \xrightarrow{\beta \mapsto \beta(e)} & F_\tau(H, V) \end{array}$$

commute. So the restriction map ϕ_V^* makes the diagram (6.40) commute.

(ii) Let $\phi : H \longrightarrow G$ and $\psi : K \longrightarrow H$ be two group homomorphism and V a G -representation. Then

$$\psi_V^* \circ \phi_V^* = (\phi \circ \psi)_V^*.$$

The composition is associative.

(iii) $\text{Id}_V^* : E(G, V) \longrightarrow E(G, V)$ is the identity map.

PROOF. It's straightforward to check by the formula (6.45) and that the restriction map of the global K-theory is associative.

(i) $R_{\phi,\tau}(\alpha)(e) = r_{\phi,\tau} \circ \alpha(e) = \text{res}|_{\Lambda_H(\tau)}^{\Lambda_G(g)} \alpha(e)$. So the diagram (6.47) commutes, which implies (6.40) commutes.

(ii) Let $\rho_g : G \longrightarrow E_g(G, V)$ be a $C_G(g)$ -equivariant map for each $g \in G^{\text{tors}}$. Note that if we have $\psi(\sigma) = \tau$ and $\phi(\tau) = g$, then $r_{\phi,\tau} \circ r_{\psi,\sigma} = r_{\phi \circ \psi, \sigma}$ since both sides are identity maps on the underlying spaces.

Then we have for any $k \in K$,

$$\begin{aligned} \psi_V^* \circ \phi_V^* \left(\prod_{g \in G^{\text{tors}}} \rho_g \right) &= \prod_g \prod_{\tau} \prod_{\sigma} r_{\psi,\sigma} \circ r_{\phi,\tau} \rho_g(\phi(\psi(k))) \\ &= \prod_g \prod_{\tau} \prod_{\sigma} r_{\psi \circ \phi, \sigma} \rho_g(\phi \circ \psi(k)) = (\phi \circ \psi)_V^* \left(\prod_{g \in G^{\text{tors}}} \rho_g \right) \end{aligned}$$

where τ goes over all the elements in H^{tors} with $\phi(\tau) = g$ and σ goes over all the elements in K^{tors} with $\psi(\sigma) = \tau$. So

$$\psi_V^* \circ \phi_V^* = (\phi \circ \psi)_V^*.$$

(iii) For the identity map $\text{Id} : G \longrightarrow G$, by the formula of the restriction map, $\text{Id}_V^* \left(\prod_{g \in G^{\text{tors}}} \rho_g \right) = \prod_{g \in G^{\text{tors}}} \rho_g$, thus, is the identity. \square

Appendix A. Join

A.1. Definition.

DEFINITION A.1. In topology, the join $A * B$ of two topological spaces A and B is defined to be the quotient space

$$(A \times B \times [0, 1]) / R,$$

where R is the equivalence relation generated by

$$(a, b_1, 0) \sim (a, b_2, 0) \text{ for all } a \in A \text{ and } b_1, b_2 \in B,$$

$$(a_1, b, 1) \sim (a_2, b, 1) \text{ for all } a_1, a_2 \in A \text{ and } b \in B.$$

At the endpoints, this collapses $A \times B \times \{0\}$ to A and $A \times B \times \{1\}$ to B .

The join $A * B$ is the homotopy colimit of the diagram (whose maps are projections)

$$A \longleftarrow A \times B \longrightarrow B.$$

A nice way to write points of $A * B$ is as formal linear combination $t_1 a + t_2 b$ with $0 \leq t_1, t_2 \leq 1$ and $t_1 + t_2 = 1$, subject to the rules $0a + 1b = b$ and $1a + 0b = a$. The coordinates correspond exactly to the points in $A * B$.

We need the topology on $A * B$ to make the four maps below continuous

$$\begin{aligned} A * B &\longrightarrow A, \quad t_1 a + t_2 b \mapsto a \\ A * B &\longrightarrow \mathbb{R}, \quad t_1 a + t_2 b \mapsto t_1 \\ A * B &\longrightarrow B, \quad t_1 a + t_2 b \mapsto b \\ A * B &\longrightarrow \mathbb{R}, \quad t_1 a + t_2 b \mapsto t_2. \end{aligned}$$

PROPOSITION A.2. Join is associative and commutative. Explicitly, $A * (B * C)$ is homeomorphic to $(A * B) * C$, and $A * B$ is homeomorphic to $B * A$.

PROOF. Consider the map $\rho_1 : A * (B * C) \longrightarrow (A * B) * C$ defined by

$$\rho_1(s_1(t_1 a + t_2 b) + s_2 c) = \begin{cases} s_1 t_1 a + (s_1 t_2 + s_2) \left(\frac{s_1 t_2}{s_1 t_2 + s_2} b + \frac{s_2}{s_1 t_2 + s_2} c \right) & \text{if } s_1 t_2 + s_2 \text{ is nonzero;} \\ 1a & \text{if } s_1 t_2 + s_2 = 0. \end{cases}$$

This is a continuous map in the topology we want. We can also define a map analogously $\rho_2 : (A * B) * C \longrightarrow A * (B * C)$ defined by

$$\rho_2(s_1 a + s_2(t_1 b + t_2 c)) = \begin{cases} (s_1 + s_2 t_1) \left(\frac{s_1}{s_1 + s_2 t_1} a + \frac{s_2 t_1}{s_1 + s_2 t_1} b \right) + s_2 t_2 c & \text{if } s_1 t_2 + s_2 \text{ is nonzero;} \\ 1c & \text{if } s_1 t_2 + s_2 = 0. \end{cases}$$

It's straightforward to check the composition $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$ are both identity maps. Thus, $A * (B * C)$ is homeomorphic to $(A * B) * C$.

Since there is a homeomorphism

$$\begin{aligned} f : A * B &\longrightarrow B * A \\ t_1 a + t_2 b &\mapsto t_2 b + t_1 a, \end{aligned}$$

join is commutative. □

A.2. Group Action on the Join.

EXAMPLE A.3. Let G be a compact Lie group. Let A, B be G -spaces. Then $A * B$ has a G -structure on it by

$$(A.1) \quad g \cdot (t_1 a + t_2 b) := t_1(g \cdot a) + t_2(g \cdot b), \text{ for any } g \in G, a \in A, b \in B, \text{ and } t_1, t_2 \geq 0, t_1 + t_2 = 1.$$

It's straightforward to check (A.1) defines a continuous group action.

EXAMPLE A.4. Let G and H be compact Lie groups. Let A be a G -space and B a H -space. Then $A * B$ has a $G \times H$ -structure on it by

$$(A.2) \quad (g, h) \cdot (t_1 a + t_2 b) := t_1(g \cdot a) + t_2(h \cdot b), \text{ for any } g \in G, a \in A, b \in B, \text{ and } t_1, t_2 \geq 0, t_1 + t_2 = 1.$$

It's straightforward to check (A.2) defines a continuous group action.

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